Consider a gas of N nonrelativistic fermions with spin 1/2 and mass m initially at zero temperature and confined in a volume V_0 and kept at.

a. Express the kinetic energy of the gas in terms of N and V_0.

b. What is the pressure of the gas? You can assume here that the gas is ideal.

c. Now the gas is allowed to expand to the volume $V_1 \gg V_0$ without any energy exchange with the outside world. Calculate the temperature of the gas after it will reach an equilibrium due to weak interactions between the fermions.

d. What is the pressure of the gas in the final state.

Define $L = V_0^{1/3}$ as the length of one side of the cube containing the gas. We can write the wavefunction of one of the fermions as:

$$\psi = c_1 e^{i(\mathbf{k}\cdot\mathbf{x})}$$

Because we are limited to volume V_0 , we must have:

$$k_x = \frac{\pi n_x}{L}; \quad k_y = \frac{\pi n_y}{L}; \quad k_z = \frac{\pi n_z}{L}$$

So that the total energy is:

$$U = \frac{\hbar^2}{2m}\sum_{i=1}^{N}\left(k_x^2 + k_y^2 + k_z^2\right) = \frac{\hbar^2\pi^2}{2mL^2}\sum_{i=1}^{N}\left(n_x^2 + n_y^2 + n_z^2\right)$$

Since T=0, the fermions are in their minimum occupancy, but since they have spin 1/2 each level can be filled by only two fermions. Thus:

$$U = \frac{6\hbar^2\pi^2}{2mL^2}\sum_{n=1}^{N/6} n^2 \approx \frac{6\hbar^2\pi^2}{2mL^2}\frac{1}{3}\left(\frac{N}{6}\right)^3 = \frac{\hbar^2\pi^2}{mV_0^{2/3}}\left(\frac{N}{6}\right)^3$$

Where we used the limit of very large N for the second equality.

We can use the ideal gas equation:

$$U = \frac{3}{2}Nk_BT$$

along with:

$$PV = Nk_BT$$

to find:

$$P = \frac{2U}{3V_0} = \frac{2\hbar^2\pi^2}{3mV_0^{5/3}}\left(\frac{N}{6}\right)^3$$

Since there can be no energy exchange, and the number is fixed, we still have:

$$U = \frac{\hbar^2\pi^2}{mV_0^{2/3}}\left(\frac{N}{6}\right)^3$$

So that:

$$T = \frac{2U}{3k_B N} = \frac{2\hbar^2\pi^2}{3mk_B N V_0^{2/3}}\left(\frac{N}{6}\right)^3$$

Again we just use the ideal gas equation:

$$P = \frac{2U}{3V_1} = \frac{2\hbar^2\pi^2}{3mV_1 V_0^{2/3}}\left(\frac{N}{6}\right)^3$$