Consider N non-interacting quantized spins in a magnetic field $\vec{B} = B\hat{z}$. The energy of the spins is $-BM_z$, where:

$$M_z \equiv \mu \sum_{i=1}^{N} S_z^{(i)}$$

is the total magnetization. For each spin, $S_z^{(i)}$ takes only 2S+1 values -S, -S+1, ..., S-1, S. Given the temperature of the system T:

a. Calculate the Gibbs partition function Z(T,B);

b. Calculate the Gibbs free energy G(T,B) and evaluate its asymptotic behavior at weak ($\mu B \gg k_B T$) magnetic field;

c. Calculate the zero-field magnetic susceptibility

$$\chi \equiv \left(\frac{\partial M_z}{\partial B}\right)_{B=0}$$

d. Calculate the magnetic susceptibility at strong fields$\mu B \gg k_B T$.

The partition function for one particle is given by:

$$Z_1 = e^{\beta E} = e^{-\beta B \mu S_z}$$

We can get the expectation of S_z by:

$$\langle S_z \rangle = \frac{\sum_{S_z} S_z e^{-\beta B \mu S_z}}{\sum_{S_z} e^{-\beta B \mu S_z}}$$

Where the sums are over all possible values of S_z. Using this, we get the total partition function to be:

$$Z(T,B) = \left(e^{-\beta B \mu \langle S_z \rangle}\right)^N = e^{-\beta B \mu N \langle S_z \rangle}$$

The Gibbs free energy may be found by:

$$G = -k_B T \ln(Z) = k_B T \beta B \mu N \langle S_z \rangle = B \mu N \langle S_z \rangle$$

In the case that $\mu BS \ll k_B T$, $e^{\beta \mu BS} \sim 1$, and so we find:

$$\langle S_z \rangle \sim \frac{\sum_{S_z} S_z}{\sum_{S_z} 1} = 0$$

so that the Gibbs free energy $G \longrightarrow 0$.

In the other case that $e^{\beta \mu BS} \gg e^{\beta \mu B(S-1)}$, so:

$$\langle S_z \rangle \sim \frac{-S e^{\beta B \mu S}}{e^{\beta B \mu S}} = -S$$

and the Gibbs free energy $G \longrightarrow -B \mu N S$.

The magnetic susceptibility is defined by:

$$\chi \equiv \left(\frac{\partial M_z}{\partial B}\right)_{B=0} = \mu N \left(\frac{\partial \langle S_z \rangle}{\partial B}\right)_{B=0}$$

Taking the derivative

$$\left(\frac{\partial\langle S_z\rangle}{\partial B}\right)_{B=0} = \frac{\sum_{S_z} -\beta\mu S_z^2}{\sum_{S_z} 1} + \frac{\sum_{S_z} S_z}{\sum_{S_z} -\beta\mu S_z} = \frac{\sum_{S_z} -\beta\mu S_z^2}{2S+1} - \frac{k_B T}{\mu}$$

Plugging in:

$$\chi = -\mu N\left(\beta\mu\frac{\sum_{S_z} S_z^2}{2S+1} + \frac{1}{\beta\mu}\right)$$

1.1.4 Part d

This is similar to the last part, but we get:

$$\left(\frac{\partial\langle S_z\rangle}{\partial B}\right)_{\mu B\gg k_B T} = \frac{-S^2\mu\beta e^{\beta B\mu S}}{e^{\beta B\mu S}} + \frac{-Se^{\beta B\mu S}}{\beta\mu S e^{\beta B\mu S}} = -S^2\mu\beta - \frac{1}{\beta\mu}$$

So that:

$$\chi = -\mu N\left(S^2\mu\beta + \frac{1}{\beta\mu}\right)$$