Jan 2004 #1 (SM)

N non-interacting, quantized spins. $\vec{B} = B\hat{z}$

$E = -BM_z$, $\quad M_z = \mu \sum_{i=1}^{N} S_z^{(i)}$

$S_z^{(i)}$ can take on values $-S, -S+1, \ldots, S-1, S$ $\quad$ ($2S+1$ values)

Given temperature T

a. Partition Function for one spin:

$Z_1 = \sum_r e^{-\beta E_r}$ $\quad$ when $S_z = m$, $\quad E_m = -\mu B m$

$$Z_1 = \sum_{m=-S}^{S} e^{\beta \mu B m} \qquad \text{define } \beta\mu B = \frac{\mu B}{kT} \equiv x$$

$$Z_1 = \sum_{m=-S}^{S} e^{xm} = \sum_{m'=0}^{2S} e^{x(m'-S)} = e^{-xS} \underbrace{\sum_{m'=0}^{2S} e^{xm'}}_{\text{Geometric Sum}}$$

$$Z_1 = e^{-xS}\left(\frac{e^{x(2S+1)}-1}{e^{x}-1}\right) = \frac{e^{x(S+1)}-e^{-xS}}{e^{x}-1} = \frac{e^{x(S+\frac{1}{2})}-e^{-x(S+\frac{1}{2})}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}$$

$$Z_1 = \sinh[x(S+\tfrac{1}{2})]/\sinh(\tfrac{x}{2})$$

Partition Function for N non-interacting spins:

$$Z = Z_1^{N}$$

$$Z = \left(\frac{\sinh[x(S+\frac{1}{2})]}{\sinh(\frac{x}{2})}\right)^{N}$$

b. $F = -kT\ln Z$ $\quad$ Helmholtz Free Energy

$$F = -kTN\left[\ln\sinh[x(S+\tfrac{1}{2})] - \ln\sinh(\tfrac{x}{2})\right] \qquad F(T,B) \qquad x = \frac{\mu B}{kT}$$

$$dF = -SdT - M_z dB$$

$$M_z = -\frac{\partial F}{\partial B} = -\frac{\partial x}{\partial B}\frac{\partial F}{\partial x} = -\frac{1}{kT}\mu\frac{\partial F}{\partial x} = \mu N\left[\frac{(S+\frac{1}{2})\cosh[x(S+\frac{1}{2})]}{\sinh[x(S+\frac{1}{2})]} - \frac{\frac{1}{2}\cosh(\frac{x}{2})}{\sinh(\frac{x}{2})}\right]$$

$$M(T,B) = \mu N\left((S+\tfrac{1}{2})\coth[x(S+\tfrac{1}{2})] - \tfrac{1}{2}\coth(\tfrac{x}{2})\right)$$

$$\cancel{G(T,B) = F(T,B) + M(T,B)\,B \quad [\text{or } G(T,B) = F(T,B) + PV = F(T,B)]}$$

weak field: $\mu BS << kT$ | $x<<1$

$$F \to -kTN\left[\ln\left(x(S+\tfrac{1}{2})\right) - \ln\frac{x}{2}\right] \longrightarrow -kTN\ln\frac{x(S+\frac{1}{2})}{x/2}$$

$$F \to -NkT\ln(2S+1)$$

strong field: $\mu B >> kT$ $\quad x>>1$ $\quad \sinh x \to \frac{e^x}{2}$

$$F \to -NkT\left[\ln\frac{e^{x(S+\frac{1}{2})}}{e^{x\cdot\frac{1}{2}}}\right] = -NkT\ln e^{xS} = -NkT\cdot xS$$

$$F \to -\mu BNS$$

c. χ at $B=0$: $\left(\frac{\partial M_z}{\partial B}\right)_{B\to 0}$

$\coth(x) \to \frac{1}{x} + \frac{1}{3}x$ for $x<<1$

for small B (or x), $M \to \mu N\left[(S+\tfrac{1}{2})\left(\frac{1}{(S+\frac{1}{2})x} + \frac{1}{3}(S+\tfrac{1}{2})x\right) - \frac{1}{2}\left(\frac{1}{\frac{1}{2}x} + \frac{1}{3}\frac{1}{2}x\right)\right]$

$$M \to \mu N\left[\frac{1}{x} + (S+\tfrac{1}{2})^2\frac{x}{3} - \frac{1}{x} - \frac{1}{12}x\right]$$

$$M \to \frac{\mu Nx}{3}\left[(S+\tfrac{1}{2})^2 - \tfrac{1}{4}\right] = \frac{\mu Nx}{3}\left[S^2 + S\right]$$

$$M \to \frac{\mu^2 NS(S+1)}{3kT}\cdot B$$

$$\chi_{B\to 0} = \frac{\mu^2 NS(S+1)}{3kT}$$

d. For large x, $\coth(x) \to 1$

$M \to$ constant

$\chi \to 0$ for large B