1 January 2004, Thermodynamics, Problem 1

1.1 (a)

$$Z_1 = \sum_{i=-S}^{S} e^{B\mu i/T} = \frac{sinh(B\mu(S+1/2)/T)}{sinh(B\mu/2T)}$$
$$Z = \frac{Z_1^N}{N!} = \frac{1}{N!}\frac{sinh^N(B\mu(S+1/2)/T)}{sinh^N(B\mu/2T)} \tag{1}$$

1.2 (b)

$$G = -TlnZ = -T\left[-lnN! + Nlnsinh(\mu B(S+1/2)/T) - Nlnsinh(B\mu/2T)\right] \tag{2}$$

$$G \approx -T\left[-NlnN + N + Nln(2S+1)\right] \quad \mu B \leq \mu BS << kT \tag{3}$$

$$G \approx -T\left[-NlnN + N + \frac{N\mu BS}{T}\right] \tag{4}$$

1.3 (c)

$$< M^i >= \mu < S^i >$$
$$< S^i >= \sum_{i=-S}^{S} \frac{ie^{B\mu i}}{Z} = \frac{1}{\mu}\frac{\partial(lnZ)}{\partial B} = coth(\mu B(S+1/2)/T)\frac{S+1/2}{T} - coth(\mu B/2T)\frac{1}{2T}$$
$$< M_z >= N < M^i >= \frac{N\mu}{T}\left[coth(\mu B(S+1/2)/T)(S+1/2) - coth(\mu B/2T)\right]$$
$$\chi = \frac{\mu N}{T}\left[\frac{-1}{sinh^2[\mu B(S+1/2)/T]}\frac{\mu(S+1/2)^2}{T} + \frac{1}{4sinh^2(\mu B/2T)}\frac{\mu}{T}\right]$$

As the field goes to zero, we must expand the sinh functions to third order in order to get the correct result:

$$\chi = \frac{\mu^2 N}{T^2}\left[\frac{-1}{[\mu B(S+1/2)/T]^2\left[1+\frac{[\mu B(S+1/2)/T]^2}{3!}\right]^2}(S+1/2)^2 + \frac{1}{4(\mu B/2T)^2\left[1+\frac{(\mu B/2T)^2}{3!}\right]^2}\right] =$$
$$\approx \frac{\mu^2 N}{T^2}(S+1/2)^2\left[\frac{-\left[1+\frac{2(\mu B/2T)^2}{3!}\right]+\left[1+\frac{2[\mu B(S+1/2)/T]^2}{3!}\right]}{[\mu B(S+1/2)/T]^2\left[1+\frac{[\mu B(S+1/2)/T]^2}{3!}\right]^2\left[1+\frac{(\mu B/2T)^2}{3!}\right]^2}\right] \approx$$
$$= \frac{\mu^2 N}{3T^2}S(S+1) \tag{5}$$

The $\approx$ sign became $=$ because the approximations were cutting higher orders of $\mu B/T$, but then that went to 0, so those terms that were cut went to 0.

1.4 (d)

We start from:

$$\chi = \frac{\mu N}{T}\left[\frac{-1}{sinh^2[\mu B(S+1/2)/T]}\frac{\mu(S+1/2)^2}{T} + \frac{1}{4sinh^2(\mu B/2T)}\frac{\mu}{T}\right]$$

and apply the approximation for large x, $\sinh x \approx e^x/2$.

$$\chi \approx \frac{\mu^2 N}{4T^2}\left[-e^{-\mu B2S/T}e^{-\mu B/T}(2S+1)^2 + e^{-\mu B/T}\right] \approx \frac{\mu^2 N}{4T^2}e^{-\mu B/T} \tag{6}$$