January 2004 QM # 3

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

a. $|\psi\rangle_0 = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad H = -\vec{\mu}\cdot\vec{B} = -\mu B \sigma_y$

$$|\psi\rangle_0 = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{2i}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$|\psi\rangle_0 = \frac{1}{\sqrt{2}}\left(|\uparrow_y\rangle - i|\downarrow_y\rangle\right)$$

$$|\psi\rangle_t = \frac{1}{\sqrt{2}}\left(e^{-iHt/\hbar}|\uparrow_y\rangle - i e^{-iHt/\hbar}|\downarrow_y\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\mu Bt/\hbar}|\uparrow_y\rangle - i e^{-i\mu Bt/\hbar}|\downarrow_y\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(e^{i\mu Bt/\hbar}\cdot\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$$

$$- i e^{-i\mu Bt/\hbar}\cdot\frac{1}{\sqrt{2}}\left(i|\uparrow\rangle + |\downarrow\rangle\right)\Big)$$

$$= \frac{1}{2}\left(e^{i\mu Bt/\hbar} + e^{-i\mu Bt/\hbar}\right)|\uparrow\rangle$$

$$- \frac{1}{2i}\left(e^{i\mu Bt/\hbar} - e^{-i\mu Bt/\hbar}\right)|\downarrow\rangle$$

$$|\psi\rangle_t = \cos\left(\frac{\mu B}{\hbar}t\right)|\uparrow\rangle - \sin\left(\frac{\mu B}{\hbar}t\right)|\downarrow\rangle$$

$$S_z|\psi\rangle_t = \frac{\hbar}{2}\cos\left(\frac{\mu B}{\hbar}t\right)|\uparrow\rangle + \frac{\hbar}{2}\sin\left(\frac{\mu B}{\hbar}t\right)|\downarrow\rangle$$

$${}_t\langle\psi|S_z|\psi\rangle_t = \frac{\hbar}{2}\cos^2\left(\frac{\mu B}{\hbar}t\right) - \frac{\hbar}{2}\sin^2\left(\frac{\mu B}{\hbar}t\right)$$

$$\langle S_z\rangle_t = \frac{\hbar}{2}\left[\cos^2\left(\frac{\mu B}{\hbar}t\right) - \sin^2\left(\frac{\mu B}{\hbar}t\right)\right]$$

b. $P(\downarrow, t=T) = \sin^2\left(\frac{\mu B}{\hbar}T\right)$

c. $P(\downarrow, t=T) = P(\uparrow, t=\frac{T}{2})P(\downarrow, t=T) + P(\downarrow, t=\frac{T}{2})P(\downarrow, t=T$

$$= \cos^2\left(\frac{\mu B}{\hbar}\frac{T}{2}\right)\sin^2\left(\frac{\mu B}{\hbar}\frac{T}{2}\right) + \sin^2\left(\frac{\mu B}{\hbar}\frac{T}{2}\right)\cos^2\left(\frac{\mu B}{\hbar}\frac{T}{2}\right)$$

$$P(\downarrow, t=T) \doteq \frac{1}{2}\sin^2\left(\frac{\mu B}{\hbar}T\right)$$

a. (continued)

$$|\psi\rangle_t = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\mu B}{\hbar}t\right) - \sin\left(\frac{\mu B}{\hbar}t\right)\right]|\uparrow_x\rangle$$

$$+ \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\mu B}{\hbar}t\right) + \sin\left(\frac{\mu B}{\hbar}t\right)\right]|\downarrow_x\rangle$$

$${}_t\langle\psi|S_x|\psi\rangle_t = \frac{1}{2}\frac{\hbar}{2}\left[\cos\left(\frac{\mu B}{\hbar}t\right) - \sin\left(\frac{\mu B}{\hbar}t\right)\right]^2$$

$$- \frac{1}{2}\frac{\hbar}{2}\left[\cos\left(\frac{\mu B}{\hbar}t\right) + \sin\left(\frac{\mu B}{\hbar}t\right)\right]^2$$

$$\langle S_x\rangle_t = -\frac{\hbar}{2}\sin\left(\frac{2\mu B}{\hbar}t\right)$$

$$\langle S_y\rangle_t = \frac{1}{2}\frac{\hbar}{2} - \frac{1}{2}\frac{\hbar}{2} = 0$$

$$\therefore \langle\vec{S}\rangle_t = \frac{\hbar}{2}\left[\cos^2\left(\frac{\mu B}{\hbar}t\right) - \sin^2\left(\frac{\mu B}{\hbar}t\right)\right]\hat{z} - \frac{\hbar}{2}\sin\left(\frac{2\mu B}{\hbar}t\right)\hat{x}$$

$$\langle\vec{S}\rangle_t = \frac{\hbar}{2}\left[\cos\left(\frac{2\mu B}{\hbar}t\right)\hat{z} - \sin\left(\frac{2\mu B}{\hbar}\right)\hat{x}\right]$$

This result shows the expectation value precessing around the y-axis which is the same result as for the classical spin.