J04Q.2

a.

The wave function of a free particle with momentum p is

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}\left[px - \frac{p^2}{2m}t\right]\right) \tag{1}$$

Using the transformation equation given:

$$\begin{aligned} \hat{\psi}(x',t) &= \psi(x,t)\exp\left(-\frac{i}{\hbar}\left[mvx - \frac{m}{2}v^2 t\right]\right) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}\left[(p-mv)(x'+vt) - \left(\frac{p^2}{2m} - \frac{mv^2}{2}\right)t\right]\right) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}\left[(p-mv)x' - \frac{1}{2m}(p^2 - 2pmv + m^2v^2)t\right]\right) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}\left[p'x' - \frac{(p')^2}{2m}t\right]\right) \end{aligned} \tag{2}$$

Which is the wave function of a free particle with momentum $p' = p - mv$ as expected.

b.

By evaluating $\int \psi^* \psi$, it's easy to determine that $|N|^2 = \sqrt{\frac{m\omega}{\pi\hbar}}$ satisfies the normalization constraint.

At time $t = 0$, the initial wave function in the primed frame is

$$\hat{\psi}(x', t=0) = N \exp\left(-\frac{m\omega}{2\hbar} x'^2 - \frac{i}{\hbar} m v x'\right) \tag{3}$$

and the final ground state wave function is

$$\hat{\psi}_0(x', t=0) = N \exp\left(-\frac{m\omega}{2\hbar} x'^2\right) \tag{4}$$

Therefore the amplitude of the system being in the ground state is

$$\begin{aligned}
\left\langle \hat{\psi}_0 \middle| \hat{\psi} \right\rangle &= \int dx' \, |N|^2 \exp\left(-\frac{m\omega}{2\hbar} x'^2\right) \exp\left(-\frac{m\omega}{2\hbar} x'^2 - \frac{i}{\hbar} m v x'\right) \\
&= |N|^2 \int dx' \exp\left(-\frac{m\omega}{\hbar} x'^2 - \frac{i}{\hbar} m v x'\right) \\
&= |N|^2 \sqrt{\frac{\pi\hbar}{m\omega}} \exp\left(-\frac{mv^2}{4\hbar\omega}\right) \\
&= \exp\left(-\frac{mv^2}{4\hbar\omega}\right)
\end{aligned} \tag{5}$$

where the second-to-last line uses the Gaussian integral result with a linear imaginary term.

The probability of finding the system in the moving ground state for $t \geq 0$ is

$$\left|\left\langle \hat{\psi}_0 \middle| \hat{\psi} \right\rangle\right|^2 = \exp\left(-\frac{mv^2}{2\hbar\omega}\right) \tag{6}$$

One thought on “J04Q.2”

December 8, 2013 at 10:50 pm

Good.

I have no important comments.