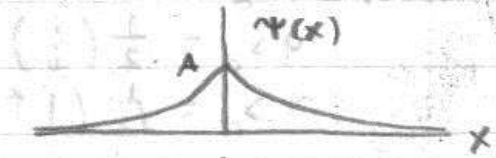
Seth Dorfman Prelims January 2, 2006

January 2004 QM

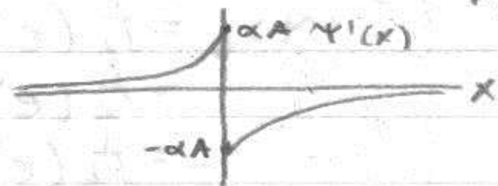
1) a. Let $\Psi(x) = Ae^{-\alpha|x|}$ $\alpha > 0$



$1 = \int_{-\infty}^{\infty} \Psi^* \Psi \, dx$

$1 = 2\int_0^{\infty} A^2 e^{-2\alpha x} dx$

$1 = 2A^2 \frac{-1}{2\alpha} e^{-2\alpha x}\Big|_{x=0}^{x=\infty}$

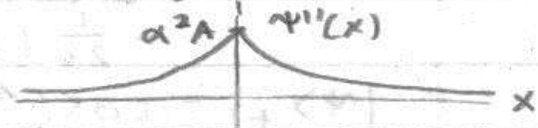


$1 = A^2 \cdot \frac{1}{\alpha}$

$A = \sqrt{\alpha}$

$\Psi'(x) = \begin{cases} \alpha A e^{\alpha x} & x<0 \\ -\alpha A e^{-\alpha x} & x>0 \end{cases}$

$E(\alpha) = \langle \Psi | H | \Psi \rangle$



$= \int_{-\infty}^{\infty} \Psi^* H \Psi \, dx$

$= \int_{-\infty}^{\infty} \Psi^* \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Psi \, dx$

$+ \int_{-\infty}^{\infty} \Psi^* \lambda V(x) \Psi \, dx$

$\Psi''(x) = \alpha^2 A e^{-\alpha|x|} = \alpha^2 \Psi(x)$

$= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \Psi^* \alpha^2 \Psi \, dx + \lambda \int_{-\infty}^{\infty} A^2 e^{-2\alpha|x|} V(x) \, dx$

$E(\alpha) = -\frac{\hbar^2}{2m}\alpha^2 + \lambda\alpha \int_{-a}^{a} V(x) e^{-2\alpha|x|} dx$

For small $\alpha << \frac{1}{a}$ $e^{-2\alpha|x|} \approx 1$ for $-a \le x \le a$

$E(\alpha) = -\frac{\hbar^2}{2m}\alpha^2 + \lambda\alpha\int_{-a}^{a} V(x)\,dx$

As long as $\int_{-a}^{a} V(x)\,dx < 0$ then $E(\alpha) < 0$ for any positive λ and the Hamiltonian has a bound state.

b. $E(\alpha)$ is a minima for large α, so use $\alpha = \frac{1}{10a}$ which is large, but still small enough that the approximation $\alpha << \frac{1}{a}$ still holds.

$E_{\text{upper bound}} = -\frac{\hbar^2}{2m}\left(\frac{1}{10a}\right)^2 + \lambda\left(\frac{1}{10a}\right)\int_{-a}^{a} V(x)\,dx$

$E_{\text{upper bound}} = \frac{-\hbar^2}{200ma^2} + \frac{\lambda}{10a}\int_{-a}^{a} V(x)\,dx$

c. No, a three dimensional problem can be reduced to a one-dimensional problem using $U(r) = rR(r)$, but this imposes the additional boundary condition $U(0) = 0$ which is true for the first excited state in one dimension. The additional boundary condition would invalidate the approximation $\int_{-a}^{a} V(x)|\Psi|^2 dx \approx \int_{-a}^{a} V(x)\,dx$ used above since $|\Psi|^2 = 0$ at $x = 0$.