J04M.3 FALLING STICK

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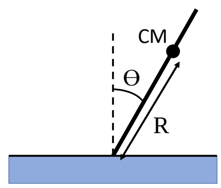

FIGURE 1

1. PROBLEM

A thin stick with some arbitrary linear mass density $\mu(x)$ along it is initially at rest. It has one end on a table and makes an angle θ_0 with the vertical. The stick-table contact point has an infinite coefficient of friction.

Let m be the total mass of the stick, R be the distance from the contact point to the center of mass, I_{CM} be the moment about the center of mass, and g be the acceleration due to gravity.

1.1. **(a.)** The stick is released from rest and allowed to fall to the table. Find the condition that the end of the stick initially in contact with the table *does* rise from the table as the stick falls. Express the condition in terms of $\theta_0, m, g, R,$ and I_{CM}.

To begin, recognize the initial energy of the system is simply:

$$U_i = mgR\cos(\theta_0).$$

Since no work is done on the stick during the entire procedure[1] we recognize that the energy at any arbitrary moment must equal U_i, and thus:

$$mgR\cos(\theta_0) = mgR\cos(\theta) + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}I_{CM}\dot{\theta}^2. \tag{1}$$

We can take a derivative of equation (1) with respect to time to obtain:

$$0 = -mgR\sin(\theta)\dot{\theta} + (mR^2 + I_{CM})\dot{\theta}\,\ddot{\theta},$$

leading to the equation:

$$\ddot{\theta} = \frac{mgR\sin(\theta)}{mR^2 + I_{CM}}. \tag{2}$$

Next, we must constrain the motion using kinematics. The vertical equation of motion is simply:

$$m\ddot{y} = N - mg,$$

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[1]Except for the work done by gravity, but we have included the potential explicitly.

where N is the normal force of the table. The trick to this question is realizing that we want to solve for the angle that allows $N = 0$, that is, the end of the stick on the table will successfully lift off at the moment the normal force becomes zero. So we simplify the equation to[2]:

$$\ddot{y} = -g.$$

We then recognize $y = R\cos(\theta)$ and take time derivatives:

$$\dot{y} = -R\sin(\theta)\dot{\theta},$$

$$\ddot{y} = -R\cos(\theta)\dot{\theta}^2 - R\sin(\theta)\ddot{\theta}.$$

Fortunately, we have already calculated what $\ddot{\theta}$ and $\dot{\theta}^2$ from energy considerations, and simply plug them in from (1) and (2) above:

$$-g = -\frac{gmR^2}{mR^2 + I_{CM}}(2(\cos(\theta_0) - \cos(\theta))\cos(\theta) + \sin^2(\theta)),$$

we simplify by defining β as $I_{CM} = \beta mR^2$,

$$1 + \beta = 2\cos(\theta_0)\cos(\theta) - 3\cos^2(\theta) + 1.$$

We obtain a quadratic equation for $\cos(x)$:

$$\cos(\theta) = \frac{1}{3}(\cos(\theta_0) \pm \sqrt{\cos^2(\theta_0) - 3\beta}).$$

This solves for the angle at which the stick lifts off of the table, so for that angle to be attainable we impose the condition[3]:

$$0 \leq \frac{1}{3}(\cos(\theta_0) \pm \sqrt{\cos^2(\theta_0) - 3\beta}) \leq \cos(\theta_0).$$

To best understand this condition, we must adjust the limits (recognizing $\cos(\theta_0) \geq 0$) to get:

$$-1 \leq \pm\sqrt{1 - 3\frac{\beta}{\cos^2(\theta_0)}} \leq 2.$$

Since the square root gets as small as 0 and as large as unity, we see that our condition is vacuously true for both roots regardless of the value of $\cos(\theta_0)$. However, we do demand that the argument be real for the comparison to make sense. Thus, our condition on θ_0 is:

$$\cos(\theta_0) \geq \sqrt{3\beta} = \sqrt{3\frac{I_{CM}}{mR^2}}.$$

[2] You may be suspicious about why the kinematics simply works this way on the center of mass with no angles involved. Convince yourself this way: at the moment the stick lifts off of the table the stick will have no force other than gravity acting on it, so it must have acceleration $-g\hat{y}$ (freefall).

[3] The upper limit is $\cos(\theta_0)$ because if $\cos(\theta)$ becomes larger than this value that means the stick rose towards vertical, which we take to be non-physical.

1.2. **(b.)** Now consider a specific mass distribution. Let the mass be uniformly distributed along the length. For what range of initial angles θ_0 will the stick eventually life off the table?

For this mass distribution, we recognize $I_{CM} = \frac{1}{12}mR^2$. Plug in the value of β to obtain:

$$\cos(\theta_0) \geq \frac{1}{2}$$

Thus, all (positive) initial angles smaller than $\frac{\pi}{3}$ will eventually lift off the table.

To put this in more relatable terms, we should use the complement to discuss the angle relative to the surface. For a uniform rod, for which your pencil is a good approximation, dropping it to the table at an angle (relative to the table) of 30^o or more will eventually have it lift off of the table[4].

For $\theta_0 = 45^o$, the stick loses contact around the angle 66.3^o (that is, 23.7^o relative to the surface). The normal force goes to zero again around 86.04^o (3.96^o relative to the surface).

1.3. **(c.)** Consider a different mass distribution: the mass is concentrated in two points of equal mass, one at either end of the stick. Now for what range of initial angles θ_0 will the stick eventually lift off the table?

Now, $I_{CM} = mR^2$, so $\beta = 1$. Plugging this in, we get:

$$\cos(\theta_0) \geq \sqrt{3} \approx 1.73$$

But, there is no real angle satisfying the constraint, so the stick will never lift off the table for this mass distribution.

[4]Recall that our system demands infinite coefficient of friction at the point of contact, which can be approximated by having the eraser side down, or putting the tip of the pencil on a large, flat eraser.