January 2004 CM #2

$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{K}{r^{n-1}} \qquad L = mr^2\dot{\theta}$$

$$\dot{r}^2 = \frac{2}{m}E - \frac{L^2}{m^2r^2} + \frac{2}{m}\frac{K}{r^{n-1}}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}E - \frac{L^2}{m^2r^2} + \frac{2}{m}\frac{K}{r^{n-1}}}$$

$$\frac{d\theta}{dt} = \frac{L}{mr^2}$$

$$\frac{dr}{d\theta} = \frac{mr^2}{L}\sqrt{\frac{2}{m}E - \frac{L^2}{m^2r^2} + \frac{2}{m}\frac{K}{r^{n-1}}} \qquad \text{Let } v = \frac{1}{r} \quad dv = \frac{-1}{r^2}dr$$

$$-\frac{dv}{d\theta}r^2 = r^2\sqrt{\frac{2mE}{L^2} - v^2 + \frac{2m}{L^2}Kv^{n-1}}$$

$$\frac{dv}{d\theta} = -\sqrt{\frac{2mE}{L^2} - v^2 + \frac{2mK}{L^2}v^{n-1}}$$

c.f. $r = \frac{1}{v} = 2a\cos\theta$ for the particle

$$\cos\theta = \frac{1}{2av}$$

$$-\sin\theta\frac{d\theta}{dv} = \frac{-1}{2av^2}$$

$$\pm\sqrt{1-\left(\frac{1}{2av}\right)^2}\frac{d\theta}{dv} = \frac{1}{2av^2}$$

$$\frac{dv}{d\theta} = \pm 2av^2\sqrt{1-\left(\frac{1}{2av}\right)^2}$$

$$\frac{dv}{d\theta} = \pm\sqrt{4a^2v^4 - v^2}$$

Comparing the two expressions for $\frac{dv}{d\theta}$:

a. $n = 5$

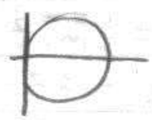
b. $\frac{2mK}{L^2} = 4a^2$

$$L^2 = \frac{mK}{2a^2}$$

$$L = \sqrt{\frac{mK}{2a^2}}$$

c. $E = 0$

d. $\frac{d\theta}{dt} = \frac{L}{m(2a\cos\theta)^2}$



one period

$\theta \Rightarrow -\frac{\pi}{2}$ to $\frac{\pi}{2}$

$$\int_0^\tau L\,dt = \int_{-\pi/2}^{\pi/2} 4ma^2\cos^2\theta\,d\theta$$

$$L\tau = 4ma^2\cdot\frac{1}{2}\pi$$

$$\frac{1}{a}\sqrt{\frac{mK}{2}}\,\tau = 2\pi ma^2$$

$$\tau = \frac{2\sqrt{2}\pi ma^3}{\sqrt{mK}}$$

$$\tau = 2\pi a^3\sqrt{\frac{2m}{K}}$$

e. $V_{eff}(r) = \frac{L^2}{2mr^2} - \frac{K}{r^4}$ $\qquad L = mV_\infty b$ ↑ impact parameter

$$\frac{dV_{eff}(r)}{dr} = \frac{-L^2}{mr^3} + \frac{4K}{r^5} = 0$$

$$\frac{1}{r^3}\left(\frac{-L^2}{m} + \frac{4K}{r^2}\right) = 0$$

$$\frac{L^2}{m} = \frac{4K}{r^2}$$

$$r^2 = \frac{4Km}{L^2} = \frac{4Km}{m^2V_\infty^2b^2} = \frac{4K}{mV_\infty^2b^2}$$

$$\sigma = \pi r^2 = \frac{4\pi K}{mV_\infty^2b^2}$$