1 January 2004, Mechanics, Problem 2

1.1 (a)

$$\dot{r} = -2asin\theta\dot{\theta}$$
$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r^{n-1}}$$
$$L = mr^2\dot{\theta}$$
$$E = \frac{1}{2}m4a^2\frac{L^2}{m^2(2acos\theta)^4} - \frac{k}{(2acos\theta)^{n-1}}$$

Since the energy is constant, it must be independent of θ. Therefore, for the two terms that depend on θ to cancel, we need for them to have the same θ-dependence. Thus,

$$n = 5 \tag{1}$$

1.2 (b)

In order for the two θ-dependent terms in the energy equation to cancel out, we need for them to add to 0:

$$2a^2\frac{L^2}{m} = k \rightarrow L = \sqrt{\frac{mk}{2a^2}} \tag{2}$$

1.3 (c)

Now that the two θ-dependent terms have cancelled, there are none left, therefore:

$$E = 0 \tag{3}$$

1.4 (d)

$$T = \int_{-\pi/2}^{\pi/2} \frac{dt}{d\theta}\,d\theta = 2\pi a^3\sqrt{\frac{2m}{k}} \tag{4}$$

1.5 (e)

Let b be the perpendicular distance between the origin and the direction of the initial velocity (this distance is called the *impact parameter*). Let d be the distance of closest approach and let v_d be the velocity of the particle at that point. Then the equation for conservation of angular momentum gives:

$$mv_\infty b = mv_d d$$

$$\frac{mv_\infty^2}{2} = \frac{mv_d^2}{2} - \frac{k}{d^4} \quad Energy\,conservation$$

Solve both equations for d to get:

$$d^2 = \frac{m(v_\infty b)^2 \pm \sqrt{m^2(v_\infty b)^4 - 8kmv_\infty^2}}{2mv_\infty^2}$$

Notice that in calculating this d, we assumed that it was greater than 0 (since, for instance, we wrote the potential energy at r=d as a finite quantity). But we are interested in finding values of b such that the distance of closest approach is exactly 0. So we need values of b for which our calculation of d leads to a contradiction, in other words, values of b for which the quantity under the square root is negative. The critical value is that for which that quantity is 0:

$$m^2(v_\infty b)^4 = 8kmv_\infty^2$$

$$b = \left(\frac{8k}{mv_\infty^2}\right)^{1/4}$$

$$\sigma = \pi b^2 = \pi \left(\frac{8k}{mv_\infty^2}\right)^{1/2} \tag{5}$$