1 January 2004, Mechanics, Problem 2

1.1  (a)

\[ \dot{r} = -2a \sin \theta \dot{\theta} \]
\[ E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{r^{n-1}} \]
\[ L = mr^2 \dot{\theta} \]
\[ E = \frac{1}{2} m 4a^2 \frac{L^2}{m^2 (2 \cos \theta)^4} - \frac{k}{(2 \cos \theta)^{n-1}} \]

Since the energy is constant, it must be independent of \( \theta \). Therefore, for the two terms that depend on \( \theta \) to cancel, we need for them to have the same \( \theta \)-dependence. Thus,

\[ n = 5 \]  \hspace{1cm} (1)

1.2  (b)

In order for the two \( \theta \)-dependent terms in the energy equation to cancel out, we need for them to add to 0:

\[ 2a^2 \frac{L^2}{m} = k \rightarrow L = \sqrt{\frac{mk}{2a^2}} \]  \hspace{1cm} (2)

1.3  (c)

Now that the two \( \theta \)-dependent terms have cancelled, there are none left, therefore:

\[ E = 0 \]  \hspace{1cm} (3)

1.4  (d)

\[ T = \int_{-\pi/2}^{\pi/2} dt \frac{d}{d\theta} \]  \hspace{1cm} (4)

1.5  (e)

Let \( b \) be the perpendicular distance between the origin and the direction of the initial velocity (this distance is called the impact parameter). Let \( d \) be the distance of closest approach and let \( v_d \) be the velocity of the particle at that point. Then the equation for conservation of angular momentum gives:

\[ mv_\infty b = mv_d d \]
\[
\frac{mv_{\infty}^2}{2} = \frac{mv_d^2}{2} - \frac{k}{d^4} \quad \text{Energy conservation}
\]

Solve both equations for \( d \) to get:

\[
d^2 = \frac{m(v_{\infty}b)^2 \pm \sqrt{m^2(v_{\infty}b)^4 - 8kmv_{\infty}^2}}{2mv_{\infty}^2}
\]

Notice that in calculating this \( d \), we assumed that it was greater than 0 (since, for instance, we wrote the potential energy at \( r=d \) as a finite quantity). But we are interested in finding values of \( b \) such that the distance of closest approach is exactly 0. So we need values of \( b \) for which our calculation of \( d \) leads to a contradiction, in other words, values of \( b \) for which the quantity under the square root is negative. The critical value is that for which that quantity is 0:

\[
m^2(v_{\infty}b)^4 = 8kmv_{\infty}^2
\]

\[
b = \left( \frac{8k}{mv_{\infty}^2} \right)^{1/4}
\]

\[
\sigma = \pi b^2 = \pi \left( \frac{8k}{mv_{\infty}^2} \right)^{1/2}
\]

\[
(5)
\]