We can define the angle that the wire makes with the horizontal by:

$$\theta = \arctan\left(4\left(\frac{r}{a}\right)^3\right)$$

Then the forces are easy to write:

$$m\ddot{r} = \left(mr\omega^2\cos\theta - mg\sin\theta\right)\cos\theta$$

Removing θ, we get $\cos\theta = a^3/\sqrt{16r^6 + a^6}$. Thus:

$$m\ddot{r} = \left(mr\omega^2 a^3 - mg4r^3\right)\frac{a^3}{16r^6 + a^6}$$

Since r>0 and a>0 , stable points exist where:

$$mr\omega^2 a^3 = mg4r^3$$

So that $r_0 = 0$ is an equilibrium point, as well as:

$$r_1 = \sqrt{\frac{\omega^2 a^3}{4g}}$$

It is easy to see that r_0 is unstable, and that r_1 is stable, but we can prove it by finding the equation of motion for $r_i + \delta r$:

$$\ddot{\delta r} = \left(\delta r\omega^2 a^3 - 12gr_i^2\delta r\right)\frac{a^3}{16r_i^6 + a^6}$$

So that for r_0:

$$\ddot{\delta r} = \omega^2\delta r$$

Which is positive so unstable, and for r_1:

$$\ddot{\delta r} = \left(\delta r\omega^2 a^3 - 12g\frac{\omega^2 a^3}{4g}\delta r\right)\frac{a^3}{16\left(\frac{\omega^2 a^3}{4g}\right)^3 + a^6} = \left(-2\omega^2 a^3\delta r\right)k$$

where k is the large fraction sticking out, which is strictly positive. Since this is negative, oscillations are stable

We already found:

$$\ddot{\delta r} = \left(-2\omega^2 a^3\delta r\right)\frac{a^3}{16\left(\frac{\omega^2 a^3}{4g}\right)^3 + a^6}$$

So that the frequency of oscillations is:

$$\omega_1 = \omega\sqrt{\frac{2}{\omega^6 a^3/(4g^3) + 1}}$$