We can define the angle that the wire makes with the horizontal by:

\[ \theta = \arctan \left( 4 \left( \frac{r}{a} \right)^3 \right) \]

Then the forces are easy to write:

\[ m\ddot{r} = \left( m r \omega^2 \cos \theta - mg \sin \theta \right) \cos \theta \]

Removing \( \theta \), we get \( \cos \theta = a^3 / \sqrt{16r^6 + a^6} \). Thus:

\[ m\ddot{r} = \left( m r \omega^2 a^3 - mg4r^3 \right) \frac{a^3}{16r^6 + a^6} \]

Since \( r>0 \) and \( a>0 \), stable points exist where:

\[ m r \omega^2 a^3 = mg4r^3 \]

So that \( r_0 = 0 \) is an equilibrium point, as well as:

\[ r_1 = \sqrt[3]{\frac{\omega^2 a^3}{4g}} \]

It is easy to see that \( r_0 \) is unstable, and that \( r_1 \) is stable, but we can prove it by finding the equation of motion for \( r_i + \delta r \):

\[ \ddot{\delta r} = \left( \delta r \omega^2 a^3 - 12gr_i^2 \delta r \right) \frac{a^3}{16r_i^6 + a^6} \]

So that for \( r_0 \):

\[ \ddot{\delta r} = \omega^2 \delta r \]

Which is positive so unstable, and for \( r_1 \):

\[ \ddot{\delta r} = \left( \delta r \omega^2 a^3 - 12g \frac{\omega^2 a^3}{4g} \delta r \right) \frac{a^3}{16 \left( \frac{\omega^2 a^3}{4g} \right)^3 + a^6} = \left( -2\omega^2 a^3 \delta r \right) k \]

where \( k \) is the large fraction sticking out, which is strictly positive. Since this is negative, oscillations are stable.

We already found:

\[ \ddot{\delta r} = \left( -2\omega^2 a^3 \delta r \right) \frac{a^3}{16 \left( \frac{\omega^2 a^3}{4g} \right)^3 + a^6} \]

So that the frequency of oscillations is:

\[ \omega_1 = \omega \sqrt{\frac{2}{\omega^6 a^3 / (4g^3) + 1}} \]