Seth Dorfman   Prelims   January 2, 2006

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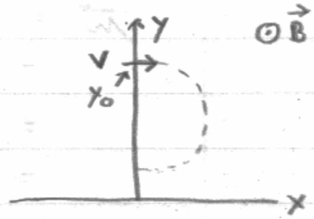
2) a. $M\dot{\vec{v}} = Q\frac{\vec{v}}{c}\times\vec{B}$ $\vec{B} = B_0\hat{z}$ $x>0$

$\dot{\vec{v}} = \frac{QB_0}{Mc}\vec{v}\times\hat{z}$ Let $\Omega = \frac{QB_0}{Mc}$



$\dot{\vec{v}} = \Omega\vec{v}\times\hat{z}$

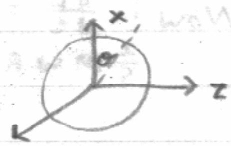
$\vec{v}(t) = v(\cos(\Omega t)\hat{x} - \sin(\Omega t)\hat{y})$

$\vec{x}(t) = y_0 + \frac{v}{\Omega}(\sin(\Omega t)\hat{x} + \cos(\Omega t)\hat{y})$

The $\hat{x}$ component is zero at $t=0, \frac{\pi}{\Omega}$

$\therefore\ t = \frac{\pi}{\Omega} = \frac{\pi Mc}{QB_0}$

b. $I_{cm} = \frac{2}{5}MR^2$



$\vec{v} = w r\sin\theta\hat{\phi}$

$\vec{L} = I w\hat{x} = \frac{2}{5}MwR^2\hat{x}$

$\vec{L} = \vec{r}\times\vec{p} = \int \rho_m(r\hat{r}\times wr\sin\theta\hat{\phi})\,d^3r$

$\vec{L} = \rho_m w\int r^2\sin\theta(\hat{r}\times\hat{\phi})\,d^3r$

$\vec{m} = \frac{1}{2c}\int \vec{r}\times\vec{J}\,d^3r$

$\vec{m} = \frac{1}{2c}\int \rho_Q(r\hat{r}\times wr\sin\theta\hat{\phi})\,d^3r$

$\vec{m} = \frac{1}{2c}\rho_Q w\int r^2\sin\theta(\hat{r}\times\hat{\phi})\,d^3r$

$\vec{m} = \frac{1}{2c}\rho_Q\frac{1}{\rho_m}\vec{L}$

$\vec{m} = \frac{1}{2c}\frac{Q}{M}\cdot\frac{2}{5}MwR^2\hat{x}$

$\vec{m} = \frac{1}{5c}QwR^2\hat{x}$, $\vec{L} = \frac{2}{5}MwR^2\hat{x}$

c. $\frac{d\vec{L}}{dt} = \vec{m}\times\vec{B}$

The angular momentum and magnetic moment precess.

Let $\hat{\ell} = \frac{\vec{m}}{|\vec{m}|} = \frac{\vec{L}}{|\vec{L}|}$

$\frac{2}{5}MwR^2\frac{d\hat{\ell}}{dt} = \frac{1}{5c}QwR^2\hat{\ell}\times B_0\hat{z}$

$\frac{d\hat{\ell}}{dt} = \frac{QB_0}{2Mc}\hat{\ell}\times\hat{z}$

$\frac{d\hat{\ell}}{dt} = \frac{\Omega}{2}\hat{\ell}\times\hat{z}$

$\Rightarrow \hat{\ell} = \cos(\frac{\Omega}{2}t)\hat{x} - \sin(\frac{\Omega}{2}t)\hat{y}$

$\hat{\ell}(t=\frac{\pi}{\Omega}) = \cos(\frac{\Omega}{2}\frac{\pi}{\Omega})\hat{x} - \sin(\frac{\Omega}{2}\frac{\pi}{\Omega})\hat{y} = -\hat{y}$

After the ball comes back to the $\vec{B}=0$ region, the angular momentum is in the $-\hat{y}$ direction,

d. Now $\vec{m} = \frac{1}{2c}\int \vec{r} \times \vec{J}\, d^3r$

$$\vec{m} = \frac{1}{2c}\int R\hat{r} \times \sigma w R\sin\theta\,\hat{\phi}\, d^3r$$

$$\vec{m} = \frac{1}{2c}\int_0^{2\pi}\int_0^{\pi} R^2\sigma w\sin\theta\,(\hat{r}\times\hat{\phi})\,R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{R^4\sigma w}{2c}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\,[-\hat{\theta}]\,d\theta\, d\phi$$

$$= \frac{R^4\sigma w}{2c}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta\,(-\cos\theta\cos\phi\,\hat{z} - \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{x})\,d\theta\, d\phi$$

$$= \frac{R^4\sigma w}{2c}\int_0^{2\pi}d\phi\int_0^{\pi}(1-\cos^2\theta)\,d(\cos\theta)\,\hat{x}$$ ($\hat{z}$ and $\hat{y}$ components cancel from ϕ-integration)

$$= \frac{R^4\sigma w}{2c}\,2\pi\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi}\hat{x}$$

$$\vec{m} = \frac{\pi R^4 w}{c}\left(\frac{Q}{4\pi R^2}\right)\frac{4}{3}\,\hat{x} = \frac{1}{3c}QwR^2\,\hat{x}$$

Now: $\frac{d\vec{L}}{dt} = \vec{m}\times\vec{B}$

$$\frac{2}{5}MwR^2\frac{d\hat{\ell}}{dt} = \frac{1}{3c}wR^2Q\,\hat{\ell}\times B_0\hat{z}$$

$$\frac{d\hat{\ell}}{dt} = \frac{1}{3}\,\frac{5}{2}\,\frac{QB_0}{Mc}\,\hat{\ell}\times\hat{z}$$

$$\frac{d\hat{\ell}}{dt} = \frac{5}{6}\Omega\,\hat{\ell}\times\hat{z}$$

$$\Rightarrow \hat{\ell} = \cos\left(\frac{5}{6}\Omega t\right)\hat{x} - \sin\left(\frac{5}{6}\Omega t\right)\hat{y}$$

$$\hat{\ell}\left(t=\frac{\pi}{\Omega}\right) = \cos\left(\frac{5}{6}\Omega\cdot\frac{\pi}{\Omega}\right)\hat{x} - \sin\left(\frac{5}{6}\Omega\frac{\pi}{\Omega}\right)\hat{y}$$

The final magnetic moment is now in the direction given by:

$$\hat{\ell}_f = \cos\left(\frac{5\pi}{6}\right)\hat{x} - \sin\left(\frac{5\pi}{6}\right)\hat{y}$$

$$\hat{\ell}_f = -\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}$$