Jan 2003 #3 (SM)

ideal Boltzmann gas, of N spin $\frac{1}{2}$ particles, thermally isolated, temp. T_i in a strong magnetic field H

For a single particle, $Z_{(1)} = Z_{spin} Z_{ideal\ gas}$ because the spin energy and translational energy are noninteracting

$$Z_{spin} \quad e^{-\beta(-\mu H)} + e^{-\beta(\mu H)} = 2\cosh(\beta\mu H)$$

$$Z_{gas}: \quad \frac{V}{h^3}\left(\frac{2\pi m}{\beta}\right)^{3/2}$$

$$Z_{(1)} = \frac{2V}{h^3}\cosh(\beta\mu H)\left(\frac{2\pi m}{\beta}\right)^{3/2} \qquad \beta = \frac{1}{kT}$$

$$Z = \frac{Z_{(1)}^N}{N!} = \frac{2^N}{h^{3N}}\frac{V^N}{N!}\cosh^N(\beta\mu H)\left(\frac{2\pi m}{\beta}\right)^{3N/2}$$

$$F = -kT\ln Z \cong -kT\left(N\ln Z_{(1)} - N\ln N + N\right)$$

$$= -NkT\left(\ln Z_{(1)} - \ln N + 1\right)$$

$$= -NkT\left(\frac{3}{2}\ln k_B T + \ln V + \ln\left[\cosh\left(\frac{\mu H}{kT}\right)\right] + \underbrace{\ln\frac{2(2\pi m)^{3/2}}{h^3} + 1 - \ln N}_{\sigma = \text{constant}}\right)$$

$$S = -\frac{\partial F}{\partial T} = Nk_B\left(\frac{3}{2}\ln k_B T + \ln V + \ln\left[\cosh\left(\frac{\mu H}{kT}\right)\right] + \sigma\right) + Nk_B T\left(\frac{3}{2k_B T}\right)$$

$$+ Nk_B T\left(\frac{-\frac{\mu H}{k}\frac{1}{T^2}\cdot\sinh\left(\frac{\mu H}{kT}\right)}{\cosh\left(\frac{\mu H}{kT}\right)}\right)$$

$$S = Nk_B\left(\frac{3}{2} + \frac{3}{2}\ln k_B T + \ln V + \ln\left[\cosh\left(\frac{\mu H}{k_B T}\right)\right] - \frac{\mu H}{kT}\tanh\left(\frac{\mu H}{kT}\right) + \sigma\right)$$

b. Thermally isolated $dQ = 0 \Rightarrow dS = 0$ (slow process; quasistatic)

$$S(T_i, H) = S(T_f, 0)$$

$$\frac{3}{2}\ln(kT_i) + \ln\left[\cosh\left(\frac{\mu H}{kT_i}\right)\right] - \frac{\mu H}{kT_i}\tanh\frac{\mu H}{kT_i} = \frac{3}{2}\ln k_B T_f + \ln 1 - 0$$

$$\frac{3}{2}\ln k_BT_f = \frac{3}{2}\ln k_BT_i + \ln(\cosh x) - x\tanh x$$

$$\ln T_f = \ln T_i + \frac{2}{3}\ln(\cosh x) - \frac{2}{3}x\tanh x$$

$$T_f = T_i(\cosh x)^{2/3}\cdot e^{-\frac{2}{3}x\tanh x}$$

c. For x small (H small), meaning a small change in T as $t\to 0$,

$$\cosh x \approx 1 + \frac{1}{2}x^2$$

$$(\cosh x)^{2/3} \approx (1+\frac{1}{2}x^2)^{2/3} \approx 1 + \frac{1}{3}x^2$$

$$e^{-\frac{2}{3}x\tanh x} \simeq 1 - \frac{2}{3}x\tanh x \approx 1 - \frac{2}{3}x^2$$

$$T_f \approx T_i\left[(1+\frac{1}{3}x^2)(1-\frac{2}{3}x^2)\right]$$

$$T_f \approx T_i\left[1-\frac{1}{3}x^2\right] \qquad T_f < T_i$$

For x large, $\cosh x \to \frac{1}{2}e^x$ $\quad (\cosh x)^{2/3} \to \frac{1}{2^{2/3}}e^{\frac{2}{3}x}$

$$\tanh x \to 1 \qquad e^{-\frac{2}{3}x\tanh x} \to e^{-\frac{2}{3}x}$$

$$T_f \to T_i \cdot 2^{-2/3}$$

(To be fully rigorous, need to show $T_f > 2^{-\frac{2}{3}}T_i$ always, in case of oscillation. To rule out this possibility, show that $\frac{dT_f}{dx}$ never changes sign)

Alternatively, $\frac{d\ln T_f}{dx}$ never changes sign.

$$\frac{d\ln T_f}{dx} = \frac{2}{3}\tanh x - \frac{2}{3}\tanh x - \frac{2}{3}x\,\mathrm{sech}^2 x$$

$$= \underbrace{-\frac{2}{3}x\,\mathrm{sech}^2 x}_{\text{always negative}}$$