* My computation for part (b) assumes that after the decay, the α spins collapse to an eigenstate; i.e. $|\psi\rangle = |2,m\rangle$ with probability $= \frac{1}{5}$, NOT $|\psi\rangle = \frac{1}{\sqrt{5}}\left(|2,2\rangle + |2,1\rangle + \dots + |2,-2\rangle\right)$

(The other way would not give $f(\phi_1,\phi_2)$ is only a function of $\phi_1 - \phi_2$, but the wording of the problem is ambiguous)

Jan 2003 #3 (QM)

a) $J_{tot} = 2 \quad J_1 = 1 \quad J_2 = 1$ total basis for total angular momentum $= 2$

$|J, m_j\rangle$:

$|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$

$|1,1\rangle$ $\vdots$

$|J_1, m_1\rangle |J_2, m_2\rangle \quad J_1 = 1 \quad J_2 = 1$

$* \; |2,2\rangle = |1,1\rangle|1,1\rangle$

Find rest of column by lowering

$J_- = J_{1-} + J_{2-}$

$$J_\pm |j,m\rangle = \underbrace{\hbar}_{\text{set } \hbar = 1} \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

[Derive from $J_\pm = J_x \pm i J_y$, $[J_x, J_y] = i\hbar J_z$]

$J_- |2,2\rangle = \sqrt{6-2}\,|2,1\rangle = 2|2,1\rangle$

$= J_{1-}|1,1\rangle|1,1\rangle + J_{2-}|1,1\rangle|1,1\rangle = \sqrt{2}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$

$* \; |2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$

$J_- |2,1\rangle = \sqrt{6}\,|2,0\rangle$

$= (J_{1-} + J_{2-})\frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right)$

$= \frac{1}{\sqrt{2}}\left[\sqrt{2}|1,-1\rangle|1,1\rangle + \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|1,1\rangle|1,-1\rangle\right]$

$* \; |2,0\rangle = \frac{1}{\sqrt{6}}\left[|1,-1\rangle|1,1\rangle + 2|1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\right]$

To get the negative states, use: $\langle j_1 m_1, j_2 m_2 | j m\rangle = (-1)^{j_1+j_2-j}\langle j_1(-m_1), j_2(-m_2) | j(-m)\rangle$

$j_1 + j_2 - j = 1+1-2 = 0$

So the coefficients for the negative states are the same as for the positive states

$* \; |2,-1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,-1\rangle + |1,-1\rangle|1,0\rangle\right)$

$* \; |2,-2\rangle = |1,-1\rangle|1,-1\rangle$

here, $|j_1 m_1\rangle|j_2 m_2\rangle \longrightarrow |\ell_1, m_1\rangle|\ell_2, m_2\rangle \longrightarrow Y_{\ell_1 m_1} Y_{\ell_2 m_2}$

b. compute $f(\theta_1, \phi_1, \theta_2, \phi_2)$ when $\theta_1 = \theta_2 = \frac{\pi}{2}$, so both α's lie in the plane

each S_z substate has equal probability (unpolarized) $= \frac{1}{5}$

Probability calculation to find density $f(\phi_1,\phi_2)$:

Let $A \equiv$ event $\phi_1 < a_1$, $\phi_2 < a_2$

$P(A) = P(A \cap m=2) + P(A \cap m=1) + \dots + P(A \cap m=-2)$ (m in $|2,m\rangle$)

$= \sum_{i=-2}^{2} P(A \cap m=i)$

$= \sum_{i=-2}^{2} P(A|m=i)\,P(m=i)$

unpolarized: each state is equally likely (this is somewhat subtle because we are using symmetry arguments to say both α particles may lie in the x-y plane)

$\Rightarrow \theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$

$f_{\Phi_1\Phi_2}(\phi_1,\phi_2) = \dfrac{\partial P(\Phi_1<\phi_1, \Phi_2<\phi_2)}{\partial\phi_1\,\partial\phi_2} = \dfrac{\partial P(A)}{\partial\phi_1\partial\phi_2}$

Now, given $m=i$, we are in the state $|2,m\rangle$

For $m=2$, for example, $|2,2\rangle = |1,1\rangle|1,1\rangle = Y_{11}(\theta_1,\phi_1)\,Y_{11}(\theta_2,\phi_2)$

In this state, $f_2(\theta_1,\phi_1,\theta_2,\phi_2) = |Y_{11}(\theta_1,\phi_1)Y_{11}(\theta_2,\phi_2)|^2$

We must know the spherical harmonics:

$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$

$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$

Note, at $\theta = \frac{\pi}{2}$, $Y_{10} = 0$, $Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\,e^{\pm i\phi}$

Because of this condition $\theta = \frac{\pi}{2}$, we must take into account 2 things.

1) Not all states $|2,m\rangle$ are actually possible. $|2,1\rangle$ and $|2,-1\rangle$ are no longer possible because they are proportional to $Y_{10}=0$. Of the 3 remaining states, I assume they each occur with equal probability $= \frac{1}{3}$

2) $Y_{\ell m}$ is a joint density in θ,ϕ. If we take θ to be a specific angle, then the distribution in ϕ is actually: $f(\phi|\Theta=\theta) = \dfrac{f_{\Theta\Phi}(\phi,\theta)}{f_\Theta(\theta)}$, where for spherical harmonics, $f(\theta,\phi) = |Y_{\ell m}(\theta,\phi)|^2$, and $f(\phi,\theta=\frac{\pi}{2}) = \frac{1}{2\pi}$ for Y_{11} for example which just says that there is no ϕ dependence; the distribution is uniform

So, removing the θ dependence and renormalizing:

$Y_{10} \rightarrow 0$

$Y_{1\pm1} \rightarrow \mp\frac{1}{\sqrt{2\pi}}\,e^{\pm i\phi}$

A similar renormalizing point shows that at $\theta = \frac{\pi}{2}$,

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left[|1,-1\rangle|1,1\rangle + 2|1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle \right]$$
$$\longrightarrow \frac{1}{\sqrt{2}}\left[|1,-1\rangle|1,1\rangle + |1,1\rangle|1,-1\rangle \right]$$

Our probability expression is $P(A) = \frac{1}{3}\left[P(A|m=2) + P(A|m=0) + P(A|m=-2) \right]$

$$f(\phi_1,\phi_2) = \frac{1}{3}\left[\underbrace{\frac{\partial P(A|m=2)}{\partial\phi_1\partial\phi_2}}_{\text{joint density for the } m=2 \text{ case } = f_{m=2}(\phi_1,\phi_2)} + \frac{\partial P(A|m=0)}{\partial\phi_1\partial\phi_2} + \frac{\partial P(A|m=-2)}{\partial\phi_1\partial\phi_2} \right]$$

calculation of $f_{m=2}(\phi_1,\phi_2)$ $\quad |2,2\rangle = |1,1\rangle|1,1\rangle \longrightarrow Y_{11}(\phi_1)Y_{11}(\phi_2)$

$$\longrightarrow +\frac{1}{2\pi}e^{i(\phi_1+\phi_2)}$$

$$f_{m=2}(\phi_1,\phi_2) = |Y_{11}(\phi_1)Y_{11}(\phi_2)|^2 = \frac{1}{4\pi^2}$$

calculation of $f_{m=-2}(\phi_1,\phi_2)$ $\quad |2,-2\rangle = |1,-1\rangle|1,-1\rangle \longrightarrow Y_{1-1}(\phi_1)Y_{1-1}(\phi_2)$

$$\longrightarrow \frac{1}{2\pi}e^{-i(\phi_1+\phi_2)}$$

$$f_{m=-2}(\phi_1,\phi_2) = \frac{1}{4\pi^2}$$

this factor of $\frac{1}{\sqrt{2}}$ needs to be used

calculation of $f_{m=0}(\phi_1,\phi_2)$ $\quad |2,0\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|1,-1\rangle|1,1\rangle + |1,1\rangle|1,-1\rangle \right]$

$$\longrightarrow \frac{1}{\sqrt{2}}\left[-\frac{1}{2\pi}e^{-i(\phi_1-\phi_2)} - \frac{1}{2\pi}e^{i(\phi_1-\phi_2)} \right] = \frac{-1}{2\pi\sqrt{2}}\left(e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)} \right)$$

$$= -\frac{1}{\sqrt{2}\pi}\cos(\phi_1-\phi_2)$$

$$\Rightarrow f_{m=0}(\phi_1,\phi_2) = \frac{1}{2\pi^2}\cos^2(\phi_1-\phi_2)$$

This is correct. $\leftarrow$ I checked this formally using $f(\phi_1,\phi_2|\theta_1=\frac{\pi}{2},\theta_2=\frac{\pi}{2}) = \frac{f(\phi_1,\phi_2,\theta_1=\frac{\pi}{2},\theta_2=\frac{\pi}{2})}{f_\Theta(\theta_1=\frac{\pi}{2},\theta_2=\frac{\pi}{2})}$

$$f(\phi_1,\phi_2) = \frac{1}{3}\left[\frac{1}{4\pi^2} + \frac{1}{4\pi^2} + \frac{1}{2\pi^2}\cos^2(\phi_1-\phi_2) \right]$$
$$= \frac{1}{6\pi^2}\left(1 + \cos^2(\phi_1-\phi_2) \right)$$

Now, $\int 1 \, d\phi_1 d\phi_2 = 4\pi^2$ and $\int \cos^2(\phi_1-\phi_2)\, d\phi_1 d\phi_2 = 2\pi^2$

- Because I have kept track of all the normalization factors so carefully, the density come out already normalized

c. Transformation from variables $(\phi_1, \phi_2) \rightarrow (\omega, \gamma)$

$$\begin{aligned} \omega &= \phi_1 - \phi_2 \\ \gamma &= \phi_2 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \phi_1 &= \omega + \gamma \\ \phi_2 &= \gamma \end{aligned}$$

reverse transform

Jacobian $J = \begin{vmatrix} \frac{\partial \phi_1}{\partial \omega} & \frac{\partial \phi_2}{\partial \omega} \\ \frac{\partial \phi_1}{\partial \gamma} & \frac{\partial \phi_2}{\partial \gamma} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

joint density of ω, γ:

$$g_{\omega\gamma}(\omega, \gamma) = |J| \cdot f_{\phi_1 \phi_2}(\underset{\phi_1}{\underset{\uparrow}{\omega+\gamma}}, \underset{\phi_2}{\underset{\uparrow}{\gamma}})$$

marginal density of ω: integrate out γ dependence:

$$f_\omega(\omega) = \int g_{\omega\gamma}(\omega, \gamma)\, d\gamma = \int_0^{2\pi} f_{\phi_1\phi_2}(\omega+\gamma, \gamma)\, d\gamma$$

$$= \int_0^{2\pi} \frac{1}{6\pi^2}\left(1 + \cos^2(\omega)\right) d\gamma$$

$$\boxed{f_\omega(\omega) = \frac{1}{3\pi}\left[1 + \cos^2\omega\right]}$$