J03Q.1 Solution

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To find the cross section first remember the helpful formula for general angular momentum states

$$\sigma = \frac{4\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L \tag{1}$$

The formula is relatively easy to remember, but the difficult part is figuring out what δ_L is. Luckily the problem gives us that only the $L = 0$ state scatters so we can just solve the Schrodinger equation with an explicit phase shift to determine δ_0. Let $u(r) = rR(r)$ where $R(r)$ are the spatial wave functions. This reduces the Schrodinger equation down to

$$-\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial r^2} + V(r)u = Eu \tag{2}$$

The solutions to this differential equation is given by

$$u(r) = \begin{cases} A\sin(k_0 r) + B\cos(k_0 r), & \text{if } r < a. \\ C\sin(kr + \delta_0), & \text{if } r > a. \end{cases} \tag{3}$$

Where

$$k_0 = \sqrt{\frac{2m}{\hbar^2}(E - V_0)} \tag{4}$$

$$k = \sqrt{\frac{2m}{\hbar^2} E} \tag{5}$$

Using the condition that $u(0) = 0$ based off of the definition of u, we get that the constant B must be zero. Then, using continuity of the wave function and its first derivative we get

$$A\sin(k_0 a) = C\sin(ka + \delta_0) \tag{6}$$

$$Ak_0 \cos(k_0 a) = Ck\cos(ka + \delta_0) \tag{7}$$

Dividing these two boundary conditions gives

$$\frac{k}{k_0}\tan(k_0 a) = \tan(ka + \delta_0) \tag{8}$$

Since $E << \hbar$, we can assume that the quantity $ka + \delta_0$ is very small, thus the previous equation reduces to

$$\frac{k}{k_0}\tan(k_0 a) \approx ka + \delta_0 \tag{9}$$

$$\Rightarrow \delta_0 \approx \frac{k}{k_0}\tan(k_0 a) - ka \tag{10}$$

Plugging this value of δ_0 into our cross section equation and ignoring the other terms as the problem says, we get

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 \tag{11}$$

$$\sigma = \frac{4\pi}{k^2}\sin^2\left(\frac{k}{k_0}\tan(k_0 a) - ka\right) \tag{12}$$

Notice that the term inside the sine is proportional to k, which is assumed to be small. Using the small angle approximation, the cross section is approximately

$$\sigma \approx 4\pi a^2\left(\frac{\tan(k_0 a)}{k_0 a} - 1\right)^2 \tag{13}$$

This should be a sufficient answer for part a) and this leads to the answer to part b). The issue with the method I've shown here is that it assumed $E - V_0$ to be a positive quantity when it determined the wave functions for $r < a$ to be a trigonometric function. However if V_0 is a large positive number this is not true and the wave function within the sphere becomes exponential. It's simple to show that redoing the calculation of the wave function using $E - V_0 < 0$ gives

$$\sigma \approx 4\pi a^2\left(\frac{\tanh(k_0' a)}{k_0' a} - 1\right)^2 \tag{14}$$

Where k_0' is given by

$$k_0' = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \tag{15}$$

Thus the limit of large positive V_0 gives

$$\sigma \approx 4\pi a^2 \lim_{k_0' \to \infty}\left(\frac{\tanh(k_0' a)}{k_0' a} - 1\right)^2 \tag{16}$$

$$\sigma \approx 4\pi a^2 \tag{17}$$

Where I made use of the fact that $\lim_{x\to\infty}\tanh x = 1$

$$\tag{18}$$