1 January 2003, Quantum Mechanics, Problem 1

1.1 (a)

The exterior solution can be written (see [1]) as:

$$\psi = A\left[\frac{sin(kr)}{kr} + f\frac{e^{ikr}}{r}\right]$$

where $k = \sqrt{2mE}$, A is a normalization constant and f is the scattering amplitude. We must match this exterior solution to the interior solution, i.e., the solution to:

$$-\frac{1}{2m}\frac{\partial^2 u}{\partial r^2} + V_0 u = Eu$$
$$u = \alpha sin(pr) \quad p = \sqrt{2m(e - V_0)}$$

The wavefunction and its derivative must be continuous at the boundary:

$$\alpha sin(pa) = A\left[\frac{sin(ka)}{k} + fe^{ika}\right] \quad Continuity$$
$$p\alpha cos(pa) = A\left[cos(ka) + ikfe^{ika}\right] \quad Derivative$$
$$\frac{1}{p}tan(pa) = \frac{\frac{sin(ka)}{k} + fe^{ika}}{cos(ka) + ike^{ika}f}$$

In the low energy limit we get:

$$f = \frac{\left[\frac{1}{p}tan(pa) - a\right](1 - ika)}{1 - \frac{ik}{p}tan(pa)}$$
$$\frac{d\sigma}{d\Omega} = |f|^2$$
$$\sigma = 4\pi\frac{d\sigma}{d\Omega} = 4\pi\frac{[tan(pa) - ap]^2}{p^2 + k^2tan^2(pa)} \tag{1}$$

1.2 (b)

One would think that the whole problem should be solved again for this case, but actually as suggested we can start from our result, using the following:

$$p = \sqrt{-2mV_0(1 - \frac{E}{V_0})} = i\sqrt{2mV_0}\left(1 - \frac{E}{2V_0}\right)$$
$$tan(i\alpha) = itanh(\alpha)$$

To zeroth order in ka, we can get:

$$\sigma = 2\pi \frac{[tanh[\sqrt{2mV_0}a] - a\sqrt{2mV_0}]^2}{mV_0 + mEtanh^2(\sqrt{2mV_0}a)} = 2\pi \frac{tanh^2[\sqrt{2mV_0}a] + a^2 2mV_0 - 2tanh[\sqrt{2mV_0}a]a\sqrt{2mV_0}}{mV_0 + mEtanh^2(\sqrt{2mV_0}a)}$$

$$\sigma \approx 2\pi \frac{tanh^2[\sqrt{2mV_0}a] + a^2 2mV_0 - 2tanh[\sqrt{2mV_0}a]a\sqrt{2mV_0}}{mV_0}$$

$$\sigma = \lim_{V_0/E \to \infty} \sigma = 2\pi 2a^2 = 4\pi a^2 \tag{2}$$

This is because $tanh$ tends to 1.

References

[1] D. Griffiths, *Introduction to Quantum Mechanics*.