1 January 2003, Quantum Mechanics, Problem 1

1.1 (a)

The exterior solution can be written (see [1]) as:
\[
\psi = A \left[ \frac{\sin(kr)}{kr} + fe^{ikr} \right]
\]

where \( k = \sqrt{2mE} \), A is a normalization constant and f is the scattering amplitude. We must match this exterior solution to the interior solution, i.e., the solution to:
\[
-\frac{1}{2m} \frac{\partial^2 u}{\partial r^2} + V_0 u = Eu
\]

\[
u = \alpha \sin(pr) \quad p = \sqrt{2m(e - V_0)}
\]

The wavefunction and its derivative must be continuous at the boundary:
\[
\alpha \sin(pa) = A \left[ \frac{\sin(ka)}{k} + fe^{ika} \right] \quad \text{Continuity}
\]
\[
p\cos(pa) = A \left[ \cos(ka) + ikfe^{ika} \right] \quad \text{Derivative}
\]
\[
\frac{1}{p} \tan(pa) = \frac{\sin(ka)}{k} + fe^{ika} \cos(ka) + ikfe^{ika} f
\]

In the low energy limit we get:
\[
f = \frac{\frac{1}{p} \tan(pa) - a}{1 - \frac{k}{p} \tan(pa)} \left( 1 - ika \right)
\]
\[
\frac{d\sigma}{d\Omega} = |f|^2
\]
\[
\sigma = 4\pi \frac{d\sigma}{d\Omega} = 4\pi \frac{[\tan(pa) - ap]^2}{p^2 + k^2 \tan^2(pa)}
\] \hspace{1cm} (1)

1.2 (b)

One would think that the whole problem should be solved again for this case, but actually as suggested we can start from our result, using the following:
\[
p = \sqrt{-2mV_0(1 - \frac{E}{V_0})} = i\sqrt{2mV_0} \left( 1 - \frac{E}{2V_0} \right)
\]
\[
\tan(i\alpha) = i\tanh(\alpha)
\]

To zeroth order in \( ka \), we can get:
\[ \sigma = 2\pi \frac{[\tanh(\sqrt{2mV_0}a) - a\sqrt{2mV_0}]^2}{mV_0 + mEtanh^2(\sqrt{2mV_0}a)} = 2\pi \frac{\tanh^2(\sqrt{2mV_0}a) + a^22mV_0 - 2\tanh(\sqrt{2mV_0}a)a\sqrt{2mV_0}}{mV_0 + mEtanh^2(\sqrt{2mV_0}a)} \]

\[ \sigma \approx 2\pi \frac{\tanh^2(\sqrt{2mV_0}a) + a^22mV_0 - 2\tanh(\sqrt{2mV_0}a)a\sqrt{2mV_0}}{mV_0} \]

\[ \sigma = \lim_{V_0/E \to \infty} \sigma = 2\pi 2a^2 = 4\pi a^2 \]  \hspace{1cm} (2)

This is because \(\tanh\) tends to 1.

**References**