J03M.3

\section{J03M3}

We start by writing the Lagrangian and the canonical momentums:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{C}{2r^2}$$

$$P_\phi = mr^2\dot{\phi}$$

$$P_r = m\dot{r}$$

Then we notice that the Lagrangian is independent of ϕ, so:

$$P_\phi = mr^2\dot{\phi} = const.$$

With Lagrangian independent of time, we have hamlitonian with energy conservation equations:

$$H = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} - \frac{C}{2r^2}$$

$$H = E = const.$$

Notice that we have three constant value: P_ϕ, E and C, which detemine the condition of our problem. We need to further analysis each of the situition. \

\subsection{$P_\phi = 0$}

$$\phi = const.$$

\subsection{$P_\phi \neq 0$}

We have $\frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi}$, then we will got:

$$\frac{dr}{d\phi} = r^2 \sqrt{(E + \frac{C}{2r^2})\frac{2m}{P_\phi^2} - \frac{1}{r^2}}$$

\subsubsection{$P_\phi^2 < Cm$}

We need to keep the square root well defined, so we require:

$$E \geq -\frac{1}{2r^2}(C - \frac{P_\phi^2}{m})$$

If $E < 0$, we will further have two solutions:

let $\frac{dr}{d\phi} = 0$, we will get:

$$r = \sqrt{(\frac{P_\phi^2}{m} - C)\,\frac{1}{2E}}$$

If $\frac{dr}{d\phi} \neq 0$, we will derive that:

$$\frac{dr}{r^2\sqrt{(E + \frac{C}{2r^2})\,\frac{2m}{P_\phi^2} - \frac{1}{r^2}}} = d\phi$$

$$\phi + D_0 = \frac{arccosh(\frac{1}{r}\sqrt{\frac{(mC/P_\phi^2 - 1)}{-2mE/P_\phi^2}})}{\sqrt{mC/P_\phi^2 - 1}}$$

$$r = \frac{1}{cosh\left[(\phi + D_0)\sqrt{mC/P_\phi^2 - 1}\right]}\sqrt{\frac{(mC/P_\phi^2 - 1)}{-2mE/P_\phi^2}}$$

If $E = 0$, we will get:

$$\frac{dr}{r\sqrt{(\frac{C}{2})\,\frac{2m}{P_\phi^2} - 1}} = d\phi$$

$$\frac{\ln r}{\sqrt{(\frac{C}{2})\,\frac{2m}{P_\phi^2} - 1}} = \phi + D_1$$

$$r = e^{\sqrt{(\frac{C}{2})\frac{2m}{P_\phi^2} - 1}\;(\phi + D_1)}$$

If $E > 0$, then $\frac{dr}{d\phi} > 0$, we will derive the equation below:

$$\frac{dr}{r^2\sqrt{(E+\frac{C}{2r^2})\frac{2m}{P_\phi^2}-\frac{1}{r^2}}}=d\phi$$

$$\phi+D_2=\frac{arcsinh(-\frac{1}{r}\sqrt{\frac{mC-P_\phi^2}{2mE}})}{\sqrt{mC/P_\phi^2-1}}$$

$$r=-\frac{1}{sinh((\phi+D_2)\sqrt{mC/P_\phi^2-1})}\sqrt{\frac{mC-P_\phi^2}{2mE}}$$

where D_2 is some undetermined constant that depends on the initial condition.

\subsubsection{$P_\phi^2=Cm$}

If $E=0$, we will get:

$$r=const.$$

and r can be any value.\\

If $E>0$, we will derive that:

$$r=-\frac{1}{\phi+D_2}\frac{l}{\sqrt{2mE}}$$

\subsubsection{$P_\phi^2>Cm$}

Once again, if we set $\frac{dr}{d\phi}=0$, we get a constant r:

$$r=\sqrt{(\frac{P_\phi^2}{m}-C)\,\frac{1}{2E}}$$

If $\frac{dr}{d\phi}\neq 0$, we further require:

$$E\geq-\frac{1}{2r^2}\,(C-\frac{P_\phi^2}{m})>0$$

Then we will get the same equation :

$$\frac{dr}{r^2\sqrt{(E+\frac{C}{2r^2})\,\frac{2m}{P_\phi^2}-\frac{1}{r^2}}}=d\phi$$

$$d\phi=\frac{d\mu}{\sqrt{\mu^2\,(\frac{mC}{P_\phi^2}-1)+\frac{2mE}{P_\phi^2}}}\qquad \mu=-1/r$$

Solve the equation, we will further get:

$$\phi+D_3=\frac{1}{\sqrt{(1-mC/P_\phi^2)}}\,arcsin(\sqrt{\frac{P_\phi^2-mC}{2mE}}\,(-\frac{1}{r}))$$

$$r=-\frac{1}{sin\left[(\phi+D_3)\sqrt{(1-mC/P_\phi^2)}\right]}\sqrt{\frac{P_\phi^2-mC}{2mE}}$$

One thought on “J03M.3”

Good, your analysis is full and you've found all the trajectories correctly.
It would make sense to also sketch them all and identify the range of the variable parametrizing the orbit in each case.