J03M.3

We are told a particle of mass m is moving in a fixed central potential characterized by $V(r) = -C/2r^2$ with an angular momentum l.
Energy conservation leads to the relation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{C}{2r^2} = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) \tag{1}$$

where $V_{eff}(r)$ is the effective potential. The two terms reduce to an effective r^{-2} potential that is either attractive, suppressed (V = 0), or repulsive when $l^2 < mC$, $l^2 = mC$, or $l^2 > mC$, respectively. The energy equation also reads:

$$-\frac{l}{m}\frac{du}{d\theta} = \sqrt{\frac{2}{m}\left(E + u^2\left(\frac{mC - l^2}{2m}\right)\right)} \tag{2}$$

by making the substitutions $dt = \frac{mr^2}{l}\,d\theta$ and $u = r^{-1}$.

Case 1 ($l^2 < mC$ and $E < 0$)

In this case, a turning point occurs where $E = V_{eff}$ such that the radial velocity is zero. From the energy equation, this quantity is $r_t = \sqrt{\frac{l^2 - mC}{2mE}}$.
We may separate variables and implicitly integrate the expression in (2):

$$-\int \frac{du}{\sqrt{\frac{2mE}{l^2} + u^2\left(\frac{mC}{l^2} - 1\right)}} = \int d\theta \tag{3}$$

where

$$-\int \frac{du}{\sqrt{\frac{2mE}{l^2}+u^2\left(\frac{mC}{l^2}-1\right)}} = -\frac{1}{\sqrt{\frac{mC}{l^2}-1}}\int \frac{du}{\sqrt{\frac{2mE}{mC-l^2}+u^2}} \qquad (4)$$

which may be solved by making a hyperbolic substitution such that $u=\sqrt{\frac{2mE}{l^2-mC}}cosh(\phi)$.

The solution to the integral on the left hand side of (3) reduces to an inverse hyperbolic cosine function:

$$\theta_0-\theta=\frac{1}{\sqrt{\frac{mC}{l^2}-1}}cosh^{-1}\left(u\sqrt{\frac{l^2-mC}{2mE}}\right) \qquad (5)$$

$$r=\sqrt{\frac{l^2-mC}{2mE}}\frac{1}{cosh\left(\sqrt{\frac{mC}{l^2}-1}\,(\theta_0-\theta)\right)} \qquad (6)$$

As the angular displacement increases, so does the hyperbolic cosine function. Therefore, the particle will approach the center of force as it continues to rotate.

Case 2($l^2 < mC$ and $E=0$)

From the energy equation:

$$-\int \frac{du}{u\sqrt{\frac{mC}{l^2}-1}} = \int d\theta \qquad (7)$$

$$-ln|u| = ln|r| = (\theta-\theta_0)\sqrt{\frac{mC}{l^2}-1} \qquad (8)$$

$$r=e^{(\theta-\theta_0)\sqrt{\frac{mC}{l^2}-1}} \qquad (9)$$

This is the orbit of a spiral where the radius varies exponentially with angular displacement.

Case 3($l^2 < mC$ and $E>0$)

The integral encountered here is similar in form to when E<0, but in this case, the solution to (3) is obtained by making a hyperbolic substitution such that $u = \sqrt{\frac{2mE}{mC-l^2}}\, sinh(\phi)$. The solution is likewise found to be a hyperbolic sine function.

$$r = \sqrt{\frac{mC - l^2}{2mE}} \frac{1}{sinh\left(\sqrt{\frac{mC}{l^2} - 1}\,(\theta_0 - \theta)\right)} \tag{10}$$

where the orbital path is unbounded and ranges from infinity down to the center of force.

$\mathbf{Case\ 4}$($l^2 = mC$)

In this case, there is no effective potential ($V_{eff} = 0$), and as usual, angular momentum remains constant. When E = 0, then the radial kinetic energy must be 0 from (2) and the particle travels in a circle. When E > 0, the radial velocity is a constant, and the particle travels in a spiral. This is the one situation where the particle could technically reach the center in finite time (when the radial velocity is both constant and pointing inwards).

$\mathbf{Case\ 5}$($l^2 > mC$ and $E > 0$)

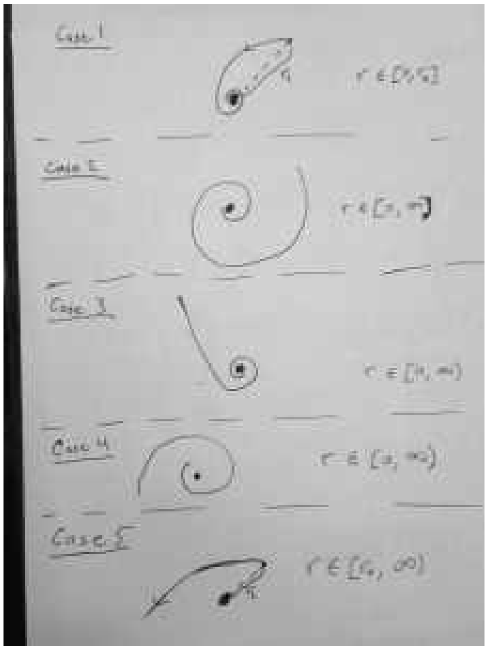
The effective potential is everywhere positive and monotonically decreasing. Therefore, the total energy must be a positive quantity, and the distance of closest approach is given by $r_t = \sqrt{\frac{l^2-mC}{2mE}}$.

In this case, the particle does not pass through the center of force because the centrifugal term is too strong. Therefore, the integral in (3) will produce a sine function by making a trigonometric substitution such that $u = \sqrt{\frac{2mE}{l^2-mC}}\, sin(\phi)$:

$$r = \sqrt{\frac{l^2 - mC}{2mE}} \frac{1}{sin\left(\sqrt{1 - \frac{mC}{l^2}}\,(\theta_0 - \theta)\right)} \tag{11}$$

This is the lone case for which the particle may not reach the center of force.

Qualitative sketches of all the orbits are shown below:



2 thoughts on “J03M.3”

Good.
Now everything seems to be correct.
Note that the Case 4 is not the only one when the particle can reach center in a finite time.

Your approach is good, you correctly identified the cases. But you should be more careful, as computational mistakes completely change the physics picture.

Please work more on this problem and correct mistakes and typos.
There are typos in (1) and (2).
Case 1: Redo computation, formula (3) is correct, (5) is not. And the corresponding sketch should be reconsidered.
Case 2: Correct. Note that the particle falls into the center.
Case 3: The formula is correct. Interpretation and sketch are not. Note that r should never be negative.
Case 4: Generally correct, but needs to be reconsidered, as some statements are not accurate. "the angular velocity do not change with distance from the center of force" -- it changes, what doesn't change is $\dot{r}$.
Case 5: Formula (10) is incorrect, but physical interpretation is correct, as if you were thinking about a different (correct) expression.

Also in fixing your solution try to clearly identify when the particle falls into the center of the force and when it doesn't (also it is useful to figure out if the time required for this is finite or not).