Jan 2003 #3 (CM)

mass m, angular momentum ℓ

$V(r) = \frac{-C}{2r^2}$

Total Energy conserved: $E = \frac{1}{2}mv^2 + V(r)$

$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$ $\quad mr^2\dot{\theta} \equiv \ell \quad \dot{\theta}^2 = \ell^2/m^2r^4$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{C}{2r^2}$$

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) \qquad V_{eff}(r) = \frac{1}{2r^2}\left(\frac{\ell^2}{m} - C\right)$$

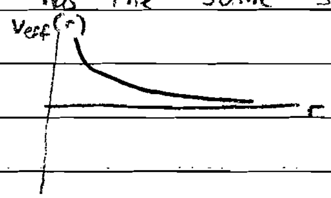
$V_{eff}(r)$ always has the same sign, and derivative, as ℓ, C are constant

① $\frac{\ell^2}{m} - C > 0$:

$\ell^2 > mC$

repulsive

for any energy

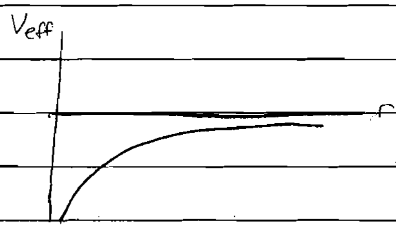


② $\frac{\ell^2}{m} - C < 0$:

$\ell^2 < mC$

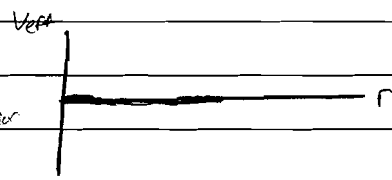
attractive

for any energy



③ $\ell^2 = mC$

$\ddot{r} = 0$; behavior depends on initial conditions



$$\frac{d}{dt}E = 0 = m\ddot{r} + \frac{dV}{dr}\frac{dr}{dt} \Rightarrow \quad m\ddot{r} = -\frac{dV}{dr}$$

Equation for orbit: $\dot{r}^2 = \frac{2}{m}\left(E - V_{eff}(r)\right)$ $\quad \dot{r} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{eff}}$

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell/mr^2}{\dot{r}} = \frac{\pm\ \ell/r^2}{\sqrt{2m(E - V_{eff})}}$$

$$\theta(r) = \int \frac{\ell/r^2\, dr}{\sqrt{2m\left[E + \frac{1}{2r^2}\left(C - \frac{\ell^2}{m}\right)\right]}} + \text{constant}$$

Let $u = \frac{\ell}{r}$ $\quad du = -\frac{\ell}{r^2} dr$

$$\theta(r) = \int \frac{-du}{\sqrt{2m}\sqrt{E + \frac{1}{2}\left(\frac{c}{\ell^2} - \frac{1}{m}\right)u^2}} + \text{const} = -\frac{1}{\sqrt{2mE}} \int \frac{du}{\sqrt{1 + \frac{1}{2mE}\left(\frac{cm}{\ell^2} - 1\right)u^2}} + \text{constant}$$

Case (1) $\ell^2 > mC$ $\quad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)$

$$\theta(r) = -\frac{1}{\sqrt{2mE}} \cdot \frac{1}{\frac{1}{\sqrt{2mE}}\sqrt{1 - \frac{mC}{\ell^2}}} \sin^{-1}\left(\frac{1}{\sqrt{2mE}}\sqrt{1 - \frac{mC}{\ell^2}}\,\frac{\ell}{r}\right) + \text{const.}$$

$$-\sqrt{1-\frac{mC}{\ell^2}}\,\theta + \phi = \sin^{-1}\left(\frac{1}{\sqrt{2mE}}\sqrt{1-\frac{mC}{\ell^2}}\,\frac{\ell}{r}\right)$$

choose $\theta = 0$ at $r = r_{min}$

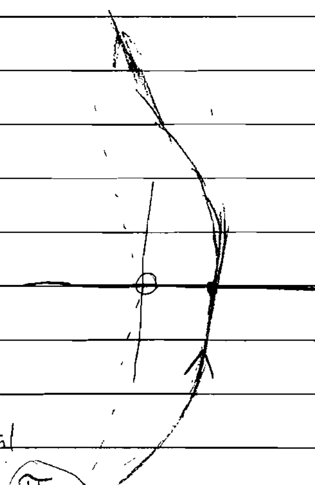
r_{min} occurs when $V_{eff}(r_{min}) = E$ $\quad E = \frac{C}{2r^2}\left(\frac{\ell^2}{mC} - 1\right) = \frac{\ell^2}{2mr^2}\left(1 - \frac{mC}{\ell^2}\right)$

$$r_{min} = \frac{\ell}{\sqrt{2mE}}\sqrt{1 - \frac{mC}{\ell^2}}$$

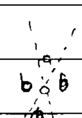
$$\phi = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\sin\left(-\sqrt{1-\frac{mC}{\ell^2}}\,\theta + \frac{\pi}{2}\right) = \frac{\sqrt{1-\frac{mC}{\ell^2}}}{\sqrt{2mE}}\,\frac{\ell}{r}$$

$$\cos\left(\sqrt{1-\frac{mC}{\ell^2}}\,\theta\right) = \sqrt{\frac{1-\frac{mC}{\ell^2}}{2mE}}\,\frac{\ell}{r}$$



$\boxed{\frac{\ell^2}{mC} > 1}$ Important result of the exact calculation which is not known from the energy plot: the total angle subtended by the orbit is $\frac{\pi}{\sqrt{1-\frac{mC}{\ell^2}}} > \pi$



$2a + 2b = 2\pi$

$2a + b = \theta_0$

deflection $= a$

$a = \theta_0 - \pi$

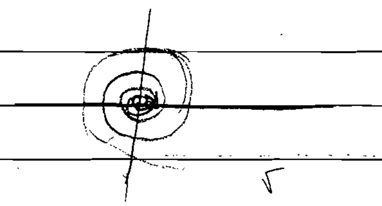
$$\text{deflection} = \theta_0 - \pi = \pi\left(\frac{1}{\sqrt{1-\frac{mC}{\ell^2}}} - 1\right)$$

Case (2) $\ell^2 < mC$ $\quad \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x)$

$$\theta(r) = \frac{-1}{\sqrt{2mE}} \cdot \frac{1}{\frac{1}{\sqrt{2mE}}\sqrt{\frac{mC}{\ell^2}-1}} \sinh^{-1}\left(\frac{\sqrt{\frac{mC}{\ell^2}-1}}{\sqrt{2mE}} \, \frac{\ell}{r} \right) + \text{const.}$$

Let $\theta = 0$ at $r=\infty$, $\sinh^{-1}(0)=0$, const $=0$

$$\Rightarrow \frac{\sqrt{\frac{mC}{\ell^2}-1}}{\sqrt{2mE}} \, \frac{\ell}{r} = - \sinh\left(\sqrt{\frac{mC}{\ell^2}-1}\, \theta \right)$$



Important result not obvious from energy plot: as $r \to 0$, $\sinh\theta \to \infty$, hence $\theta \to \infty$: infinite spiraling into $r=0$
(of course not true if body collides at some nonzero r)

case (3): $\ell^2 = mC$ $\quad V_{eff}=0 \quad E \geq 0$

$$\theta(r) = \frac{\ell}{\sqrt{2mE}} \int \frac{dr}{r^2} + \text{constant} = \frac{-\ell}{\sqrt{2mE}\, r} + \phi$$

Behavior depends on initial conditions

if $E=0$, $\dot{r}=0$ for all time - circular orbit

if $E>0$, $\dot{r}^2>0$ for all time; either moving to $r=0$, or $r=\infty$ with $\theta(r)$ as above

$E=0$ $\qquad$ $E>0$, $\dot{r}>0$ or $\dot{r}<0$