1 January 2003, Mechanics, Problem 3

We start by writing the energy and angular momentum conservation equations:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{C}{2r^2}$$
$$l = mr^2\dot{\phi}$$

Notice the solution $r = 0$ is not valid because of the potential term. Therefore, we can write:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\frac{l^2}{m^2r^4}) - \frac{C}{2r^2}$$

We then start the cases:

1.1 $l = 0$

$$\phi = const.$$

1.2 $l \neq 0$

Rewrite $\dot{r}$ as $\frac{dr}{d\phi}\dot{\phi}$, and denote $r' = \frac{dr}{d\phi}$:

$$E = \frac{1}{2}\frac{l^2}{mr^4}(r'^2 + r^2) - \frac{C}{2r^2}$$
$$r' = r^2\sqrt{\left(E + \frac{C}{2r^2}\right)\frac{2m}{l^2} - \frac{1}{r^2}}$$

1.2.1 $E < 0$

Notice that in order for the square root to remain defined, we need to impose the condition:

$$Cm > l^2$$

This admits two solutions. The first one is easy: make the thing under the square root equal 0; then you get a constant-r solution:

$$r = \sqrt{\left(\frac{l^2}{m} - C\right)\frac{1}{2E}}$$

For the second solution, assume the square root is *not* zero, and write:

$$\frac{dr}{r^2\sqrt{\left(E + \frac{C}{2r^2}\right)\frac{2m}{l^2} - \frac{1}{r^2}}} = d\phi$$

$$\phi + D = \frac{arccosh\left(\frac{1}{r}\sqrt{\frac{(mC/l^2-1)}{-2mE/l^2}}\right)}{\sqrt{mC/l^2-1}}$$

$$r = \frac{1}{cosh\left[(\phi + D)\sqrt{mC/l^2-1}\right]}\sqrt{\frac{(mC/l^2-1)}{-2mE/l^2}}$$

where D is some undetermined constant that depends on the value of the energy and angular momentum.

1.2.2 $E = 0$

Again, in order for the square root to remain defined, we need to impose:

$$Cm \geq l^2$$

And again two solutions are admitted. For the first one we set the thing under the square root to 0, and we get:

$$r = const \quad Cm = l^2$$

Notice that any r works. For the second solution, assume the square root is not zero and write:

$$\frac{dr}{r\sqrt{\left(\frac{C}{2}\right)\frac{2m}{l^2} - 1}} = d\phi$$

$$\frac{\ln r}{\sqrt{\left(\frac{C}{2}\right)\frac{2m}{l^2} - 1}} = \phi + D$$

$$r = e^{\sqrt{\left(\frac{C}{2}\right)\frac{2m}{l^2}-1}(\phi + D)} \quad Cm > l^2$$

1.2.3 $E > 0$

Once again, if we set the square root to 0, we get a solution with constant r:

$$r = \sqrt{\left(\frac{l^2}{m} - C\right)\frac{1}{2E}} \quad l^2 > Cm$$

When the square root is not 0, we once again write:

$$\frac{dr}{r^2\sqrt{\left(E + \frac{C}{2r^2}\right)\frac{2m}{l^2} - \frac{1}{r^2}}} = d\phi$$

$$d\phi = \frac{d\mu}{\sqrt{\mu^2\left(\frac{mC}{l^2} - 1\right) + \frac{2mE}{l^2}}} \quad \mu = -1/r$$

Case 1: $l^2 < mC$

$$\phi + D = \frac{arcsinh\left(-\frac{1}{r}\sqrt{\frac{mC-l^2}{2mE}}\right)}{\sqrt{mC/l^2 - 1}}$$

$$r = -\frac{1}{sinh\left((\phi + D)\sqrt{mC/l^2 - 1}\right)}\sqrt{\frac{mC - l^2}{2mE}}$$

Case 2: $l^2 = mC$

$$r = -\frac{1}{\phi + D}\frac{l}{\sqrt{2mE}}$$

Case 3: $l^2 > mC$

$$\phi + D = \frac{1}{\sqrt{(1 - mC/l^2)}} arcsin\left(\sqrt{\frac{l^2 - mC}{2mE}}\left(-\frac{1}{r}\right)\right)$$

$$r = -\frac{1}{sin\left[(\phi + D)\sqrt{(1 - mC/l^2)}\right]}\sqrt{\frac{l^2 - mC}{2mE}}$$