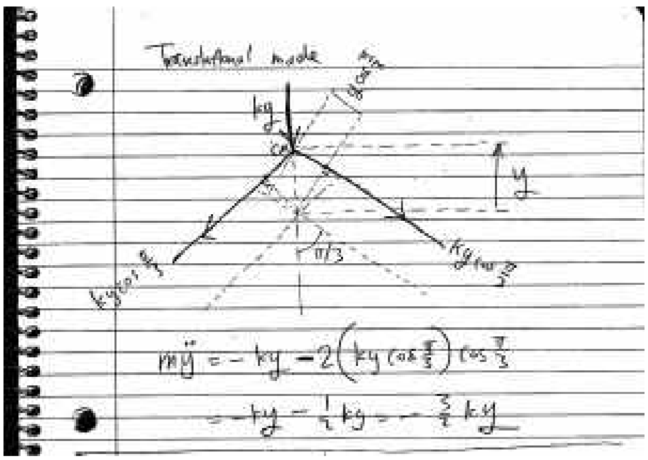
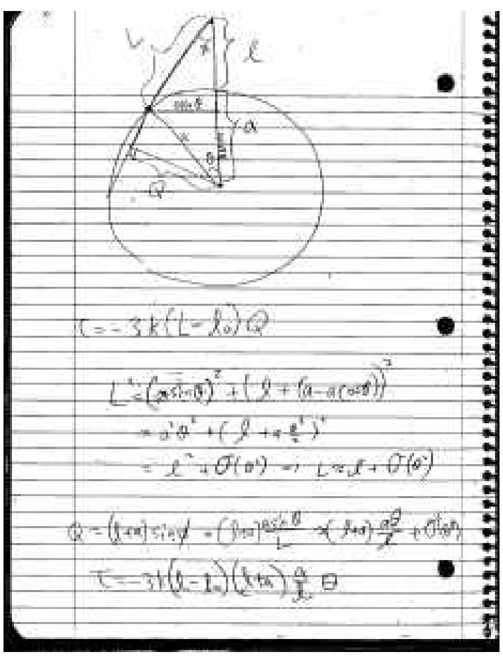
J03M.2





Solution to J03M.2 — Disk with Three Springs

We take the y-axis to point up, the x-axis to point to the right, and the z-axis to point out from the page. We place the origin of our axes at the center of the equilateral triangle. We let k denote the spring constant of the springs.

The disk has three degrees of freedom: 2 for translations along the plane, and 1 for rotations about any axis perpendicular the plane. Hence, there must be three linearly independent normal modes.

Intuitively, the simplest translational normal modes involve oscillating longitudinally along one of the springs. They should all have the same frequency due to the symmetry of the setup. However, only two of these are linearly independent. This gives us the translational normal modes. One can also imagine oscillatory rotations about the z-axis. This gives us the rotational normal mode.

We now calculate the frequency for the translational modes. We consider oscillations along the y-axis for $|y| \ll a$. The equation of motion to lowest order in y is

$$m\,\frac{d^2y}{dt^2} = -ky - 2ky\cos^2\frac{\pi}{3}$$

On the right hand side of the equation, the first term is the force coming from the upper spring, while the second term is the force coming from the two lower springs. It follows that $\frac{d^2y}{dt^2} + \omega_1^2 y = 0$, where $\omega_1 = \sqrt{\frac{3k}{2m}}$, which gives us the angular frequency for the two translational modes.

We now calculate the frequency for the rotational mode. We let θ denote the angle of rotation, with $\theta \ll 1$. The equation of motion to lowest order in θ is

$$\tau = I\,\frac{d^2\theta}{dt^2} = -3k(l-l_0)(l+a)\,\frac{a}{l}\,\theta$$

where τ denotes the z-component of the torque applied on the disk by the springs, and I denotes the moment of inertia of the disk about its axis of symmetry. It follows that $\frac{d^2\theta}{dt^2} + \omega_2^2\theta = 0$, where $\omega_2 = \sqrt{\frac{3k(l-l_0)(l+a)a}{Il}}$ is the angular frequency for the rotational mode.

Furthermore, we need

$$I = \int_{\text{disk}} dm\, r^2 = \frac{m}{\pi a^2}\int_0^a (2\pi r\, dr) r^2 = \frac{1}{2}ma^2$$

which gives us $\omega_2 = \sqrt{\frac{6k(l-l_0)(l+a)}{mal}}$.

3 thoughts on “J03M.2”

Looks correct now.

October 8, 2013 at 2:09 am

I just have realized that your expression for torque is not correct. It is correct only in the limit $l \gg a$. Please reconsider that part of your solution.

September 29, 2013 at 7:24 pm

Everything is correct.
Maybe picture could be helpful to expand explanations of certain details (especially $\cos^2 \frac{\pi}{3}$).