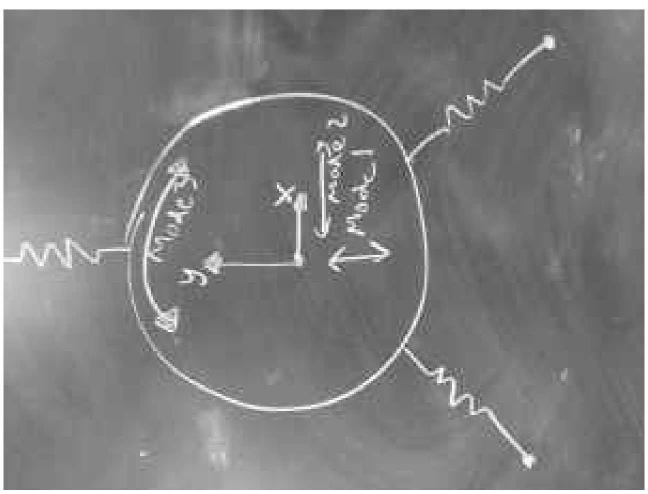
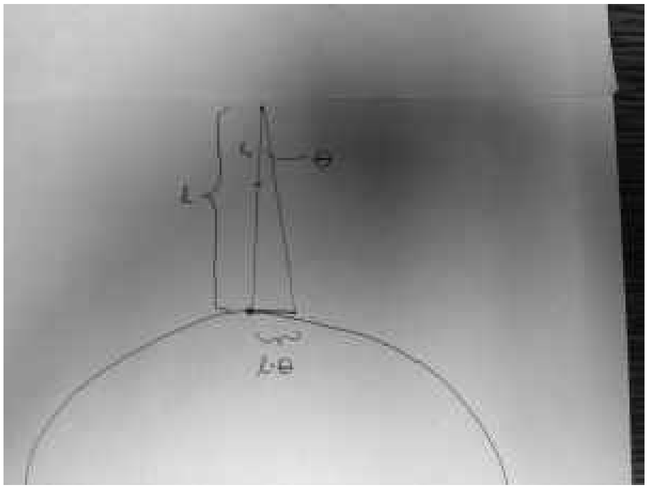
J03M.2

Solution to J03M.2 — Disk with Three Springs

There are two translational normal modes for the disk translating along the x and y axes and an additional mode in which the disk is rotating about its center (no translation).



One of the normal mode of oscillations involves the disk rotating about its center of mass with no translation. For very small θ, the displacement will be proportional to $l\theta$ (see figure below).



For rotating objects contained in a plane, Newton's second law takes the form

$$\tau = I\ddot{\theta} \tag{1}$$

where τ is the torque (where $\tau = \vec{r} \times \vec{F}$), I is the moment of inertia of the rotating body and $\ddot{\theta}$ is the tangential acceleration. The torque produced by the top spring on the disk will be

$$\tau = a(-k(l - l_0)\theta) \tag{2}$$

where a is the given radius of the disk. By symmetry, the torque exerted by all three springs will be equal in magnitude and direction. The equation of motion for the disk then becomes

$$-3k(l - l_0)a\theta = \frac{1}{2}ma^2\ddot{\theta} \Rightarrow \ddot{\theta} + \frac{6k(l - l_0)}{ma}\theta = 0 \tag{3}$$

And so we find that the frequency of oscillations for the rotating normal mode is

$$\omega = \sqrt{\frac{6k(l - l_0)}{ma}} \tag{4}$$

Since the problem states that the oscillations are small, our Taylor series expansion for the force becomes the familiar form of Hooke's Law

$$F(y) = F_0 + y\left(\frac{dF}{dy}\right)_0 + \frac{1}{2!}y^2\left(\frac{d^2F}{dy^2}\right)_0 + \ldots \Rightarrow \vec{F} = -k\vec{y} \tag{5}$$

To find the frequency of oscillation for the vertical normal mode, calculate the force from each of the springs by a displacement purely in the y-direction (Δy). The resulting force in the top spring will oppose the forces in the bottom springs (and by symmetry, the x-components of force in the bottom springs will cancel each other out). The expression for total force from the three springs will become

$$F = m\ddot{y} = -ky - 2ky\cos^2\left(\frac{\pi}{3}\right) \rightarrow \ddot{y} + \frac{3}{2}\frac{k}{m}y = 0 \tag{6}$$

for which we use the standard solution of

$$y = Ae^{i\omega t} \tag{7}$$

where

$$\omega^2 = \frac{3k}{2m} \tag{8}$$

By symmetry, the other translational mode will oscillate with the same characteristic frequency.

2 thoughts on “J03M.2”

October 8, 2013 at 4:33 pm

Check (2). In (8) include missed $\cos(\pi/3)$ and expand up to linear order in y.

September 29, 2013 at 7:06 pm

You correctly identify the normal modes, except that the three symmetry-related modes are not independent: there are only two translational degrees of freedom on the plane, and so these three modes are dependent.The idea on how to find the frequencies is correct.
However there are some technical flaws:

Equation (2) gives you the change in the tension of a single bottom spring. Then you should add forces from the two bottom springs (remembering that they are not pointing in the vertical direction). Since the angle between the forces is $\frac{2\pi}{3}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$, you'll find that the answer is different.

In equation (9) you assume that the tension of the spring is kl, while actually it's $k(l - l_0)$. And then indeed, the torque will be $-k(l-l_0)a\theta$.