J03M.2

Solution to J03M.2 — Disk with Three Springs

I'll label the coordinates of the origins of each spring s_1, s_2, s_3 , and the coordinates of the attachment points of the spring to the disk for arbitrary θ, x, y as a_1, a_2, a_3 :

$$s_1 = -(a+l)(\sqrt{3}\hat{x} + \hat{y})/2$$
$$s_2 = (a+l)(\sqrt{3}\hat{x} - \hat{y})/2$$
$$s_3 = (a+l)\hat{y}$$
$$a_1 = (x + (-\sqrt{3}a\cos\theta + a\sin\theta)/2)\hat{x} + (y + (-\sqrt{3}a\sin\theta + a\cos\theta)/2)\hat{y}$$
$$a_2 = (x + (\sqrt{3}a\cos\theta + a\sin\theta)/2)\hat{x} + (y + (\sqrt{3}a\sin\theta - a\cos\theta)/2)\hat{y}$$
$$a_3 = (x - a\sin\theta)\hat{x} + (y + a\cos\theta)\hat{y}$$

I note that the moment of inertia for a disk is $ma^2/2$. We now have all the ingredients for the Lagrangian:

$$L = (m/2)(\dot{x}^2 + \dot{y}^2 + a^2\dot{\theta}^2/2) - (k/2)((|a_1 - s_1| - l_0)^2 + (|a_2 - s_2| - l_0)^2$$
$$+ (|a_3 - s_3| - l_0)^2)$$

It is important to first put in the full equations, and THEN take the small oscillations approximations, otherwise subtle terms will be dropped. The result is:

$$(m/2)(\dot{x}^2 + \dot{y}^2 + a^2\dot{\theta}^2/2) - (3k/2)(x^2 + y^2 + (\tfrac{l-l_0}{l})(a(a+l)\theta^2))$$

Thus: $T_{ij} = \frac{1}{2}\frac{d^2T}{d\dot{q}_i d\dot{q}_j}\Big|_{\dot{q}_i, \dot{q}_j = 0} =$

$$\begin{pmatrix} m/2 & 0 & 0 \\ 0 & m/2 & 0 \\ 0 & 0 & ma^2/4 \end{pmatrix}$$

and $V_{ij} = \frac{1}{2}\frac{d^2V}{dq_i dq_j}\Big|_{q_i, q_j = 0} =$

$$\begin{pmatrix} \frac{3k}{2} & 0 & 0 \\ 0 & \frac{3k}{2} & 0 \\ 0 & 0 & \frac{3ka^2}{2}\frac{l-l_0}{l}(a+l)(a) \end{pmatrix}$$

Thus trivially solving $|V - \omega^2 T| = 0$ we get $\omega_1 = \omega_2 = \sqrt{\frac{3k}{m}}$ and $\omega_3 = \sqrt{\frac{6k}{m}\frac{l-l_0}{l}\frac{l+a}{a}}$, which correspond to three trivial eigenvectors one in each direction. The modes are thus vibrations in the $\hat{x}, \hat{y}, \hat{\theta}$ directions, respectively. This makes sense, as the symmetry of the problem favors no direction, so vibrations should be either purely in one spatial direction or purely rotations around the axis of the disk.

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2 thoughts on “J03M.2”

October 27, 2013 at 2:01 am

Good. I like your answers now.

October 8, 2013 at 2:20 am

This is a good approach, since by using the general apparatus you obtain the answer almost for free. You just have to be careful with calculations.
There is still some mistake in computations. Take a look at your expression for ω_3. First, it doesn't depend on m, which is probably a typo, since dimensionality is wrong in such case. More subtle: your expression doesn't depend on a. However, it is clear that if $a \to 0$, torque will be proportional to a, while moment of inertia will be proportional to a^2. Thus the frequency ω_3 should diverge as $a \to 0$. So there should be a dependence on a.

Please check your computations.

Also small typo: lhs of your 8th equation should be a_2.

Terminological remark: coordinates are numbers, while your s_i and a_i are vectors rather than coordinates.