1 January 2003, Mechanics, Problem 1

1.1 (a)

$$V = -V_0\Theta(z)$$
$$\nabla V = -V_0\delta(z)\hat{z}$$
$$\boldsymbol{F} = V_0\delta(z)\hat{z} = m\ddot{\boldsymbol{x}}$$

This means that the z-component of the momentum is not conserved, but the component perpendicular to the z-direction is. Thus:

$$v_i sin\theta_1 = v_f sin\theta_2$$
$$V_0 = \frac{m}{2}(v_f^2 cos^2\theta_2 - v_i^2 cos^2\theta_1)$$
$$v_i^2 = \frac{2E}{m}$$

Solving for $tan\theta_2$:

$$tan\theta_2 = \frac{\sqrt{E}sin\theta_1}{\sqrt{V_0 + Ecos^2\theta_1}}$$
$$sin\theta_2 = sin\theta_1\sqrt{\frac{E}{E+V_0}} \quad (1)$$

1.2 (b)

Now there are two deflections: upon entering and leaving the sphere. This part requires some geometry, and I can't draw pictures... Let θ_1 and θ_2 be defined as above, and let θ_3 be the angle that the outcoming particle makes with respect to the normal to the sphere. By symmetry, we must have:

$$\theta_3 = \theta_1$$

Let θ be the angle that the outgoing particle makes with respect to the incoming particle. Some analysis of the geometry shows that:

$$2\theta_1 - 2\theta_2 = \theta$$

Finally, let b be the impact parameter, as explained in the problem. Some more geometry shows that:

$$b = asin\theta_1$$

That's it for the geometry. Now it's all messy algebra:

$$b = a sin\left(\frac{\theta}{2} + \theta_2\right) = a\left[sin\left(\frac{\theta}{2}\right)cos(\theta_2) + cos\left(\frac{\theta}{2}\right)sin(\theta_2)\right]$$

$$b = a\left[sin\left(\frac{\theta}{2}\right)\sqrt{1 - sin^2\theta_1\left(\frac{E}{E+V_0}\right)} + cos\left(\frac{\theta}{2}\right)sin\theta_1\sqrt{\frac{E}{E+V_0}}\right]$$

$$b = \sin\left(\frac{\theta}{2}\right)\sqrt{a^2 - b^2\left(\frac{E}{E+V_0}\right)} + cos\left(\frac{\theta}{2}\right)b\sqrt{\frac{E}{E+V_0}}$$

Solve for b to obtain:

$$b = \frac{a\sin(\theta/2)}{1 - 2cos(\theta/2)\sqrt{\frac{E}{E+V_0}} + \frac{E}{E+V_0}}$$

Take the derivative with respect to θ and then plug into the expression:

$$\frac{d\sigma}{d\Omega} = \frac{\frac{a^2}{2}\left[\frac{cos(\theta/2)}{2}\left(\frac{V_0}{E+V_0}\right) - \sqrt{\frac{E}{E+V_0}}\right]}{cos(\theta/2)\left[1 - 2cos(\theta/2)\sqrt{\frac{E}{E+V_0}} + \frac{E}{E+V_0}\right]^3} \quad (2)$$