January 2003 Preliminary Exam, Electromagnetism Problem 3

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Problem:

A rod of length L and negligible cross sectional area carries a total charge Q uniformly distributed along its length. It is rotated slowly with $\omega \ll c/L$ in the x-y plane as shown.

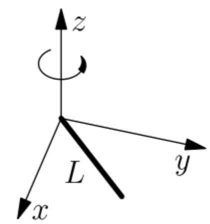

Figure 1: *Figure from J03E.3*

(a) What are the electric dipole moment and rate at which electric dipole energy is radiated?

(b) What are the magnetic dipole moment and rate at which magnetic dipole energy is radiated?

Solution:

(a) The electric dipole moment can be found by integrating the dipole moments caused by each infinitely small portion of the charged rod:

$$dq = \frac{Q}{L} dl \tag{1}$$

$$d\vec{p} = dql\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right) \tag{2}$$

$$\implies \vec{p}(t) = \frac{Q}{L}\int_0^L l\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right) dl = \frac{QL}{2}\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right) \tag{3}$$

I'll need the second derivative of this moment later:

$$\implies \ddot{\vec{p}}(t) = \frac{-QL\omega^2}{2}\left(\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}\right) \tag{4}$$

$$\left\langle |\ddot{\vec{p}}(t)|^2 \right\rangle = \frac{Q^2L^2\omega^4}{4}\left\langle \cos^2(\omega t) + \sin^2(\omega t)\right\rangle = \frac{Q^2L^2\omega^4}{4} \tag{5}$$

From David Tong's notes on EM (The radiation chapter was SO helpful for my prelim prep), I can cite the magnetic potential from an electric dipole and derive the power from there:

$$\vec{A}_{ED}(\vec{r},t) = \frac{\mu_0}{4\pi r}\dot{\vec{p}}(\vec{r}, t - \frac{r}{c}) \tag{6}$$

$$\vec{B}(\vec{r},t) = \nabla \times \vec{A}(\vec{r},t) = \frac{-\mu_0}{4\pi rc}\,\hat{r} \times \ddot{\vec{p}}(\vec{r}, t - \frac{r}{c}) \tag{7}$$

$$\left\langle \vec{S} \right\rangle = \frac{1}{\mu_0}\left\langle \vec{E} \times \vec{B} \right\rangle = \frac{c}{\mu_0}\left\langle |\vec{B}|^2 \right\rangle \hat{r} = \frac{\mu_0}{16\pi^2 r^2 c}\left\langle |\hat{r} \times \ddot{\vec{p}}(\vec{r}, t - \frac{r}{c})|^2 \right\rangle \hat{r} = \frac{\mu_0 \left\langle |\ddot{\vec{p}}(\vec{r}, t - \frac{r}{c})|^2 \right\rangle}{16\pi^2 r^2 c}\sin^2(\theta)\hat{r} \tag{8}$$

$$\implies P_{ED} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin(\theta) \frac{\mu_0 \left\langle |\ddot{\vec{p}}(\vec{r}, t - \frac{r}{c})|^2 \right\rangle}{16\pi^2 r^2 c} \sin^2(\theta) = \frac{\mu_0 \left\langle |\ddot{\vec{p}}(\vec{r}, t - \frac{r}{c})|^2 \right\rangle}{8\pi c} \int_0^{\pi} d\theta \sin^3(\theta) \tag{9}$$

$$\implies P_{ED} = \frac{\mu_0 \left\langle |\ddot{\vec{p}}(\vec{r}, t - \frac{r}{c})|^2 \right\rangle}{6\pi c} = \frac{\mu_0}{6\pi c} \frac{Q^2 L^2 \omega^4}{4} = \frac{\mu_0 Q^2 L^2 \omega^4}{24\pi c} \tag{10}$$

(b) You could find the magnetic moment by finding how much current each charge dq produces and the area of the loop it sweeps out, but it'll be faster to use the gyromagnetic ratio here, given that the rod is uniformly charged. The moment of inertia of the rod about the farthest point can be found using the parallel axis theorem:

$$I = I_{CM} + MD^2 = \frac{1}{12} ML^2 + M \left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2 \tag{11}$$

$$\vec{m} = \gamma \vec{L} = \gamma I \vec{\omega} = \left(\frac{Q}{2M}\right) \left(\frac{1}{3} ML^2\right) (\omega \hat{z}) = \frac{QL^2 \omega}{6} \hat{z} \tag{12}$$

Again, from David Tong's notes, the magnetic vector potential for a magnetic dipole is the following:

$$\vec{A}_{MD}(\vec{r}, t) = \frac{-\mu_0}{4\pi r c} \hat{r} \times \dot{\vec{m}}(\vec{r}, t - \frac{r}{c}) \tag{13}$$

The magnetic dipole moment is time independent, so the vector potential is zero. This means that the power radiated by the dipole moment is zero, which makes sense because the rotating rod looks essentially like a constant current loop.