1 J03E.2

1.1 a

These two charges have following potential

$$V(x) = \frac{Q}{|\bar{x} - \bar{b}|} - \frac{\sqrt{\frac{a}{b}}Q}{|\bar{x} - \bar{a}|}, \tag{1}$$

where $\bar{a}$ and $\bar{b}$ are vector-positions of charges. Let us expand this formula and calculate scalar products

$$\begin{aligned} V(x) = & \frac{Q}{\sqrt{r^2 - 2rb\cos\phi + b^2}} - \frac{Q}{\sqrt{\frac{b}{a}}\sqrt{r^2 - 2ra\cos\phi + a^2}} = \\ & Q\left(\frac{1}{\sqrt{r^2 - 2rb\cos\phi + b^2}} - \frac{1}{\sqrt{\frac{br^2}{a} - 2rb\cos\phi + ab}}\right). \end{aligned} \tag{2}$$

Here $r = |x|$, we see that $r = \sqrt{ab}$ makes $V(x) = 0$ irrespective of the value of angle ϕ. Hence we have an equipotential spherical surface $V(x) = 0$ of radius $R = \sqrt{ab}$. It is worth noting that position a can be obtained from b by inversion $a = \frac{R^2}{b}$, the a charge is $q = -\frac{R}{b}Q$.

1.2 b

Electrical potential in this problem can be obtained by method of images, a partial justification of which was given in previous section. In order to make $\phi(r, \theta\, z)$ potential equal to zero on a sphere of radius R we introduce an inverse conjugate charge $-q' = -\frac{R}{b}q$ at $(\frac{R^2}{b^2}r_0, 0, \frac{R^2}{b^2}z_0)$. Next we have to make $\phi(r, \theta\, z)$ on the plane $z = 0$. In order to achieve that we add to mirrored opposite charges $-q$ and q' at positions $(r_0, 0, -z_0)$ and $(\frac{R^2}{b^2}r_0, 0, -\frac{R^2}{b^2}z_0)$ respectively. A final potential is a sum of one real charge and three images.

$$\begin{aligned} \phi(r, \theta\, z) = & \frac{q}{\sqrt{(r\cos\theta - r_0)^2 + r^2\sin^2\theta + (z - z_0)^2}} - \\ & \frac{q}{\sqrt{(r\cos\theta - r_0)^2 + r^2\sin^2\theta + (z + z_0)^2}} - \\ & \frac{\frac{R}{b}q}{\sqrt{(r\cos\theta - \frac{R^2}{b^2}r_0)^2 + r^2\sin^2\theta + (z - \frac{R^2}{b^2}z_0)^2}} + \\ & \frac{\frac{R}{b}q}{\sqrt{(r\cos\theta - \frac{R^2}{b^2}r_0)^2 + r^2\sin^2\theta + (z + \frac{R^2}{b^2}z_0)^2}}. \end{aligned} \tag{3}$$

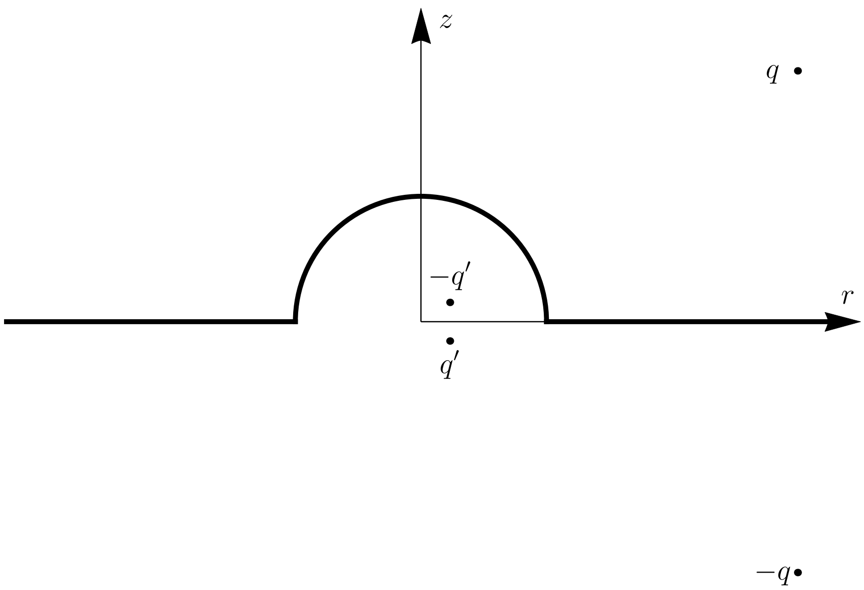

Figure 1: Real charge and its images

1.3 c

Applying Coulomb law directly we get

$$F = -\frac{\frac{R}{z_0}q^2}{\left(z_0 - \frac{R^2}{z_0}\right)^2} + \frac{\frac{R}{z_0}q^2}{\left(z_0 + \frac{R^2}{z_0}\right)^2} - \frac{q^2}{4z_0^2}. \tag{4}$$