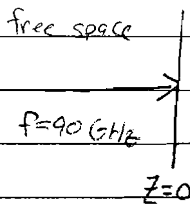
Jan 2003 #1 (EM)



Aluminum ($\mu \approx 1$, $\sigma = 3.5 \cdot 10^{17}\, s^{-1}$)

$\epsilon = ?$

a. In aluminum, $\vec{E} = E_0 e^{i(kz-\omega t)} \hat{x}$

Dispersion Relation: $\rho_f = 0$ in conductor: $\nabla \cdot \vec{E} = \frac{4\pi\rho_f}{\epsilon} = 0$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\mu \vec{J} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}$$

Ohm's Law: $\vec{J}_f = \sigma \vec{E}$: $\nabla \times \vec{B} = \frac{4\pi\mu\sigma}{c}\vec{E} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}$

$$\nabla \times (\nabla \times \vec{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \vec{B} = -\frac{1}{c^2}\left(4\pi\mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

$= \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ $\quad (\nabla \cdot \vec{E} = 0)$: transverse wave

$$\Rightarrow \quad \nabla^2 \vec{E} = \frac{1}{c^2}\left(4\pi\mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

$$-k^2 = -\frac{\mu\epsilon}{c^2}\omega^2 - \frac{i4\pi\mu\sigma\omega}{c^2}$$

$$k^2 = \frac{\epsilon\mu}{c^2}\omega^2\left(1 + \frac{4\pi\sigma i}{\omega\epsilon}\right)$$

$k = k_r + ik_i \qquad k^2 = k_r^2 - k_i^2 + i\,2k_r k_i$

$$k_r^2 - k_i^2 = \frac{\epsilon\mu\omega^2}{c^2}$$

$$k_r k_i = \frac{2\pi\mu\sigma\omega}{c^2} \qquad k_i = \frac{1}{k_r}\frac{2\pi\mu\sigma\omega}{c^2}$$

$$k_r^2 - \frac{1}{k_r^2}\left(\frac{2\pi\mu\sigma\omega}{c^2}\right)^2 = \frac{\epsilon\mu\omega^2}{c^2}$$

$$k_r^4 - \frac{\epsilon\mu\omega^2}{c^2}k_r^2 - \left(\frac{2\pi\mu\sigma\omega}{c^2}\right)^2 = 0$$

$$k_r^2 = \frac{\frac{\epsilon\mu\omega^2}{c^2} \pm \sqrt{\left(\frac{\epsilon\mu\omega^2}{c^2}\right)^2 + 4\left(\frac{2\pi\mu\sigma\omega}{c^2}\right)^2}}{2}$$

$$k_r^2 = \frac{\epsilon\mu\omega^2}{c^2}\left[\frac{1 \pm \sqrt{1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2}}{2}\right] \qquad k_r^2 > 0, \text{ choose } + \text{ sign}$$

$$k_r = \frac{\sqrt{\epsilon\mu}\,\omega}{c}\left[\frac{\sqrt{1 + \left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} + 1}{2}\right]^{1/2}$$

$$k_i^2 = k_r^2 - \frac{\epsilon\mu\omega^2}{c^2}$$

$$k_i = \sqrt{\epsilon\mu}\,\frac{\omega}{c}\left[\frac{\sqrt{1+\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2} - 1}{2}\right]^{1/2}$$

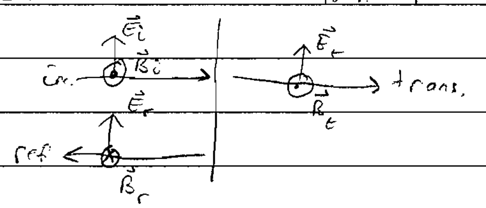
$k_r(\omega)$, $k_i(\omega)$

Limit of a good conductor, $\sigma \to \infty$

$$k^2 \cong i\,\frac{4\pi\mu\sigma\omega}{c^2} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k \approx (1+i)\cdot\sqrt{\frac{2\pi\mu\sigma\omega}{c^2}} \quad \text{or} \quad \approx \sqrt{\epsilon\mu}\,\frac{\omega}{c}\sqrt{\frac{4\pi\sigma}{\omega\epsilon}}\,\frac{1+i}{\sqrt{2}}$$

b. fraction of incident power reflected



$$\left(\frac{\vec B_{\parallel}^{(1)}}{\mu_1} - \frac{1}{\mu_2}\vec B_{\parallel}^{(2)} = \vec K_f \times \hat n\right) \quad \text{but } \sigma E \ne \infty,\ K_f = 0$$

$$BC\text{'s} \cdot \epsilon_1 E_{\perp}^{(1)} = \epsilon_2 E_{\perp}^{(2)} \qquad E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$$

$$B_{\perp}^{(1)} = B_{\perp}^{(2)} \qquad \frac{1}{\mu_1}B_{\parallel}^{(1)} = \frac{1}{\mu_2}B_{\parallel}^{(2)}$$

$$\nabla\times\vec E = -\frac{1}{c}\frac{\partial\vec B}{\partial t} \qquad \vec K\times\vec E = \frac{\omega}{c}\vec B \qquad |B| = |E|\cdot\sqrt{\epsilon\mu}$$

$\epsilon_1 = 1$ $\qquad \epsilon_2 \Rightarrow \epsilon\left(1 + \frac{4\pi\sigma i}{\omega\epsilon}\right)$ effective dielectric constant $(\epsilon = 1?)$

$$\epsilon_2 \approx i\,\frac{4\pi\sigma}{\omega}$$

Derivation of reflected power:

$$E_i + E_r = E_t$$

$$\frac{1}{\mu_1}(B_i - B_r) = \frac{1}{\mu_2}B_t \qquad B = \sqrt{\epsilon\mu}\,E \qquad \rightarrow \qquad \sqrt{\frac{\epsilon_1}{\mu_1}}(E_i - E_r) = \sqrt{\frac{\epsilon_2}{\mu_2}}E_t$$

$$E_t = E_i + E_r$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}E_i - \sqrt{\frac{\epsilon_1}{\mu_1}}E_r = \sqrt{\frac{\epsilon_2}{\mu_2}}E_i + \sqrt{\frac{\epsilon_2}{\mu_2}}E_r$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\frac{\epsilon_1}{\mu_1}} - \sqrt{\frac{\epsilon_2}{\mu_2}}}{\sqrt{\frac{\epsilon_1}{\mu_1}} + \sqrt{\frac{\epsilon_2}{\mu_2}}}$$

$$\mu_1 = \mu_2 = 1 \qquad \epsilon_1 = 1$$

$$\frac{E_r}{E_i} = \frac{1 - \sqrt{\epsilon_2}}{1 + \sqrt{\epsilon_2}}$$

Reflected Power $\langle S_{ref}\rangle$ $\qquad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$

Incident Power $\langle S_{inc}\rangle$

$\langle \vec{S}\rangle = \frac{c}{8\pi}\vec{E}\times\vec{H}^*$ Proof: For two complex quantities of the same frequency,

$$\langle u v\rangle \Rightarrow \langle \mathrm{Re}\,u\ \mathrm{Re}\,v\rangle = \left\langle \frac{u+u^*}{2}\ \frac{v+v^*}{2}\right\rangle = \frac{1}{4}\langle uv + u^*v + uv^* + u^*v^*\rangle$$

$$= \frac{1}{4}\langle uv^* + u^*v\rangle = \frac{1}{4}(uv^* + u^*v) + \frac{1}{2}\mathrm{Re}(uv^*)$$

$$\vec{H}^* = \frac{1}{\mu}\vec{B}^* = \frac{1}{\mu}\sqrt{\epsilon\mu}\ \hat{n}\times\vec{E}^* = \sqrt{\frac{\epsilon}{\mu}}\ \hat{n}\times\vec{E}^*$$

$$\langle\vec{S}\rangle = \frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\ \vec{E}\times(\hat{n}\times\vec{E}^*) = \frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\ \hat{n}\ |\vec{E}_0|^2$$

$$\frac{\langle S_{ref}\rangle}{\langle S_{inc}\rangle} = \frac{\frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\ |E_r|^2}{\frac{c}{8\pi}\sqrt{\frac{\epsilon_1}{\mu_1}}\ |E_i|^2} = \frac{|E_r|^2}{|E_i|^2} = \left|\frac{E_r}{E_i}\right|^2$$

$$= \left|\frac{1-\sqrt{\epsilon_2}}{1+\sqrt{\epsilon_2}}\right|^2 = \left(\frac{1-\sqrt{\epsilon_2}}{1+\sqrt{\epsilon_2}}\right)\left(\frac{1-\sqrt{\epsilon_2}^*}{1+\sqrt{\epsilon_2}^*}\right) = \frac{1-(\sqrt{\epsilon_2}+\sqrt{\epsilon_2}^*)+|\epsilon_2|}{1+(\sqrt{\epsilon_2}+\sqrt{\epsilon_2}^*)+|\epsilon_2|}$$

$$R = \frac{1-2\cdot\mathrm{Re}(\sqrt{\epsilon_2})+|\epsilon_2|}{1+2\cdot\mathrm{Re}(\sqrt{\epsilon_2})+|\epsilon_2|}$$

$$\epsilon_2 \cong i\frac{4\pi\sigma}{\omega} \qquad |\epsilon_2| = \frac{4\pi\sigma}{\omega}$$

$$\sqrt{\epsilon_2} \approx \frac{1+i}{\sqrt{2}}\sqrt{\frac{4\pi\sigma}{\omega}} = (1+i)\sqrt{\frac{2\pi\sigma}{\omega}} \qquad \mathrm{Re}(\sqrt{\epsilon_2}) \cong \sqrt{\frac{2\pi\sigma}{\omega}}$$

$$R = \frac{1-\sqrt{\frac{8\pi\sigma}{\omega}}+\frac{4\pi\sigma}{\omega}}{1+\sqrt{\frac{8\pi\sigma}{\omega}}+\frac{4\pi\sigma}{\omega}} = \frac{1-\sqrt{\frac{\omega}{2\pi\sigma}}+\frac{\omega}{4\pi\sigma}}{1+\sqrt{\frac{\omega}{2\pi\sigma}}+\frac{\omega}{4\pi\sigma}} \approx \frac{1-\sqrt{\frac{\omega}{2\pi\sigma}}}{1+\sqrt{\frac{\omega}{2\pi\sigma}}}$$

$$R \approx \left(1-\sqrt{\frac{\omega}{2\pi\sigma}}\right)\left(1-\sqrt{\frac{\omega}{2\pi\sigma}}\right)$$

$$R = 1-2\sqrt{\frac{\omega}{2\pi\sigma}} + \mathcal{O}\left(\frac{\omega}{\sigma}\right)$$

$$R = 1-\sqrt{\frac{2\omega}{\pi\sigma}} + \mathcal{O}\left(\frac{\omega}{\sigma}\right)$$

c. emissivity = absorptivity for a body in thermal equilibrium

$$T = 1-R = \sqrt{\frac{2\omega}{\pi\sigma}} + \mathcal{O}\left(\frac{\omega}{\sigma}\right)$$ = power transmitted into aluminum and absorbed

$$\frac{2\omega}{\pi\sigma} = \frac{2\cdot 2\pi\cdot 90\cdot 10^9}{\pi\cdot 3.5\cdot 10^{17}} = \frac{4\cdot 90}{3.5\cdot 10^8} = \frac{36}{3.5\cdot 10^7} \cong 10^{-6}$$

$$T = \sqrt{\frac{2\omega}{\pi\sigma}} = 10^{-3}$$

$$\boxed{\epsilon = 10^{-3}}$$