Jan 2002 #3 (SM)

$N >> 1$ oscillators at ω, M quanta

$$W(M) = \frac{(M+N-1)!}{M!(N-1)!}$$

a) $E = \frac{1}{2}N\hbar\omega + M\hbar\omega \longrightarrow \bar{E} = M\hbar\omega$ ignoring 0-point energy

$$S = k\ln W = k\left[\ln(M+N-1)! - \ln M! - \ln(N-1)!\right]$$

$$S \approx k\left[(M+N-1)\ln(M+N-1) - (M+N-1) - M\ln M + M - (N-1)\ln(N-1) + N-1\right]$$

$$S = k\left[(M+N-1)\ln(M+N-1) - M\ln M - (N-1)\ln(N-1)\right]$$

b. equilibrium with heat reservoir at temp T

$\rightarrow$ The Helmholtz free energy $F = \bar{E} - TS$ is minimized

$$F = M\hbar\omega - kT\left[(M+N-1)\ln(M+N-1) - M\ln M - (N-1)\ln(N-1)\right]$$

Number of quanta M varies to minimize F

$$\frac{\partial F}{\partial M} = \hbar\omega - kT\left[\ln(M+N-1) + 1 - \ln M - 1\right] = 0$$

$$\ln\left(\frac{M+N-1}{M}\right) = \frac{\hbar\omega}{kT} \qquad \frac{M+N-1}{M} = e^{\frac{\hbar\omega}{kT}}$$

$$M\left(e^{\frac{\hbar\omega}{kT}} - 1\right) = N-1 \approx N$$

$$\langle M \rangle = \frac{N}{e^{\frac{\hbar\omega}{kT}} - 1}$$

$$n(T) = \frac{\langle M \rangle}{N} = \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}$$

c. $C_V = \frac{\partial \bar{E}}{\partial T}$ $\qquad \bar{E} = M\hbar\omega = N\hbar\omega \cdot \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1}$

$$C_V = \frac{N\hbar\omega \cdot}{\left(e^{\frac{\hbar\omega}{kT}} - 1\right)^2} (-1) \cdot e^{\frac{\hbar\omega}{kT}} \cdot \frac{\hbar\omega}{k} \cdot \left(-\frac{1}{T^2}\right)$$

$$C_V = Nk\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\frac{\hbar\omega}{kT}}}{\left(e^{\frac{\hbar\omega}{kT}} - 1\right)^2}$$

d. $\langle S \rangle = S|_{M=\langle M \rangle}$

$$\frac{d\langle S \rangle}{dE} = \frac{d\langle M \rangle}{dE}\frac{d\langle S \rangle}{d\langle M \rangle} = \frac{1}{\hbar\omega}\frac{d\langle S \rangle}{d\langle M \rangle} =$$

$$\frac{k}{\hbar\omega}\cdot\left[\ln(\langle M \rangle + N - 1) - \ln\langle M \rangle\right] = \frac{k}{\hbar\omega}\cdot\ln\left(\frac{\langle M \rangle + N - 1}{\langle M \rangle}\right) = \frac{k}{\hbar\omega}\ln\left(e^{\frac{\hbar\omega}{kT}}\right)$$

$$= \frac{k}{\hbar\omega}\cdot\frac{\hbar\omega}{kT} = \frac{1}{T}$$

e M quanta = "balls" to divide in N oscillators = boxes, without requiring at least 1 quanta in each oscillator. N-1 partitions

M+N-1 total things which can be rearranged in any order.

Choose the location of the M quanta:

$$W = \binom{M+N-1}{M} = \frac{(M+N-1)!}{M!\,(N-1)!}$$