J02Q.3

Solution to J02Q.3 — Interacting Particles on a Line

Part a

First change variables to the center of mass frame

$$\bar{x} = \frac{x_1 + x_2}{2} \qquad (1)$$
$$x = x_1 - x_2$$

Making the Hamiltonian

$$H = \frac{p^2 + 4\bar{p}^2}{m} + (2\hbar^2 - S_T^2)U_0(x) \qquad (2)$$

In the center of mass frame $\bar{p} = 0$ so we consider just the Hamiltonian

$$H = \frac{p^2}{m} + (2\hbar^2 - S_T^2)U_0(x) \qquad (3)$$

We can separate the wavefunction into position and spin parts

$$\psi = \psi_x \psi_{S_A} \psi_{S_B} \qquad (4)$$

Since the Hamiltonian is only concerned with S_T^2 , the eigenfunctions for the spin wavefunction will be the standard triplet-singlet states for two spin-1/2 particles. Written in the S,m basis they are:

$$\psi_{S_A}\psi_{S_B} = |1,1\rangle, |1,1\rangle, |1,-1\rangle, |0,0\rangle \qquad (5)$$

The Hamiltonian now splits into two cases: one for $S = 1$, where $S_T^2 = \hbar^2(S)(S+1) = 2\hbar^2$ and another for $S = 0$, where $S_T^2 = 0$.

$$H(S=1) = \frac{p^2}{m} \tag{6}$$
$$H(S=0) = \frac{p^2}{m} + 2\hbar^2 U_0(x)$$

The solution to the S=1 Hamiltonian trivially that of a free particle

$$\psi_x(S=1) = Ce^{ik_1 x} \tag{7}$$

$$E(S=1) = \frac{\hbar^2 k_1^2}{m} \tag{8}$$

Where k_1 can take any real value. Thus when S=1 we do not get a bound state.

For the S=0 Hamiltonian we solve the Schrodinger equation:

$$H(S=0)\psi_x = \frac{p^2}{m}\psi_x + 2\hbar^2 U_0(x)\psi_x$$
$$E\psi_x = -\frac{\hbar^2}{m}\frac{\partial^2\psi_x}{\partial x^2} + 2\hbar^2(-\frac{\pi^2}{4ma^2})\psi_x$$
$$\frac{\partial^2\psi_x}{\partial x^2} = -\left(\frac{mE}{\hbar^2} + \frac{\pi^2}{2a^2}\right)\psi_x$$

The solution to this, using the condition imposed by the potential that $\psi_x = 0$ at $x = a$, is simply

$$\psi_x(S=0) = D\sin(\omega_0(x-a)) \tag{9}$$

Where

$$\omega_0^2 = \left(\frac{mE}{\hbar^2} + \frac{\pi^2}{2a^2}\right) \tag{10}$$

The potential also imposes the constraint $\psi_x = 0$ at $x = -a$. Plugging this into our solution gives the relation $\omega_0 = \frac{n\pi}{2a}$. We can now plug this into equation (7) and solve for the energy.

$$E_n(S=0) = \frac{\hbar^2\pi^2(n^2-2)}{4ma^2} \tag{11}$$

The ground state energy (n=1) is:

$$\boxed{E_{n=1} = \frac{-\hbar^2\pi^2}{4ma^2}} \tag{12}$$

Finally, normalizing the wave function gives $D = \sqrt{\frac{2}{\pi a}}$, leaving us with

$$\psi_x(S=0) = \sqrt{\frac{2}{\pi a}}\sin(\frac{\pi(x-a)}{2a}) \tag{13}$$

Part b

For a magnetic field $B_z = B_0 \cos k(x-ct)$ the perturbation Hamiltonian is

$$H' = \mu \boldsymbol{B}(x_1)\cdot \boldsymbol{S_A} + \mu \boldsymbol{B}(x_2)\cdot \boldsymbol{S_B}$$
$$H' = \mu B_z(x_1)S_{A_z} + \mu B_z(x_2)S_{B_z}$$
$$H' = \mu B_z(\tfrac{x_1+x_2}{2} + \tfrac{x_1-x_2}{2})S_{A_z} + \mu B_z(\tfrac{x_1+x_2}{2} - \tfrac{x_1-x_2}{2})S_{B_z}$$

We are given $ka \ll 1$. Since $|x_1 - x_2|$ is less than a, this implies that $k\frac{x_1-x_2}{2} \ll 1$, so we can Taylor expand the above expression to get

$$H' = \mu B_z(\frac{x_1+x_2}{2})S_{T_z} + \frac{x_1-x_2}{2}\mu B_z'(\frac{x_1+x_2}{2})(S_{A_z} - S_{B_z}) \tag{14}$$

From part (a), the ground state had S=0. The spin wave function corresponding to this is

$$\psi_{S_A}\psi_{S_B} = |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{15}$$

Apply the operators in our perturbation to this eigenstate:

$$S_{T_z}|0,0\rangle = 0 \tag{16}$$

$$(S_{A_z} - S_{B_z})\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = |1,0\rangle \tag{17}$$

So we can drop the first term in the Hamiltonian, and we know that the bound state can only transition into the $|1,0\rangle$ state. To compute this probability we use time-dependent perturbation theory.

$$c_1(t) = -\frac{i}{\hbar}\int_0^t dt' e^{i\frac{E(S=1)-E_0}{\hbar}t'}\langle\psi_x(S=1)|\frac{x_1-x_2}{2}\mu B_z'(\frac{x_1+x_2}{2})|\psi_x(S=0)\rangle \tag{18}$$

One thought on “J02Q.3”

December 15, 2013 at 10:32 pm

You're on the right track, try to finish your solution.
There are some typos and small mistakes in what you've written so far.
In (2) you have the wrong factor before $\bar{p}^2$ (not important for the solution though). In (7) forgot the i factor. In (10) and in the equation before it you've lost the factor of 2, so that actually $\omega_0^2 = \left(\frac{mE}{\hbar^2} + \frac{\pi^2}{2a^2}\right)$ and the ground state energy is negative. Everything else seems to be fine.