1 January 2002, Quantum Mechanics, Problem 2

1.1 (a)

My first guess is that one has to shift the position by -vt, and the momentum by -mv. Doing that gives:

$$\hat{\psi}(x,t) = \psi(x+vt,t)e^{-imvx}$$

where the hat denotes the wavefunction in the moving frame. Plugging this into Schrdinger's equation gives:

$$i\dot{\hat{\psi}} = ie^{-imvx}\left(\psi' v + \dot{\psi}\right) = e^{-imvx}\left(i\psi' v - \frac{1}{2m}\psi'' + V\psi\right)$$

$$H\hat{\psi} = -\frac{1}{2m}\frac{\partial^2}{\partial x^2}\left(\psi(x+vt,t)e^{-imvx}\right) + V\psi(x+vt,t)e^{-imvx} =$$

$$= \left(-\frac{1}{2m}\psi'' + iv\psi' + \frac{mv^2}{2}\psi\right)e^{-imvx} + V\psi e^{-imvx}$$

You can see that $i\dot{\hat{\psi}} \neq H\hat{\psi}$ as it stands, but the difference is only in one term that looks like the kinetic energy due to motion. It can be fixed easily, and the final answer is:

$$\hat{\psi}(x,t) = \psi(x+vt,t)e^{-imvx}e^{-imv^2t/2} \tag{1}$$

and the interpretation is that we have to shift the position, boost the momentum and boost the energy in order to get the right answer.

1.2 (b)

In the lab frame, the initial wavefunction is:

$$\psi_{bef} = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$$

where a_0 is the Bohr radius. After the boost, the lab frame moves with velocity $-v$ in a given direction with respect to the hydrogen atom. Therefore, the wavefunction afterwards is:

$$\psi_{aft} = \psi^{CM}(\boldsymbol{r} - vt\hat{z}, t)e^{imvz}e^{-imv^2t/2}$$

where I have defined z to be the direction of the boost. Assuming the boost is immediate, the wavefunction remains the same immediately afterwards as it was immediately before. Thus, the probability that the function remains in the ground state can be computed at t=0, by using as the center-of-mass final wavefunction the wavefunction of the ground state of the hydrogen atom:

$$P = |<\psi_{bef}|\psi_{aft,ground}>|^2 = \left|\int_0^\pi \int_0^\infty \frac{e^{-2r/a_0}}{\pi a_0^3} e^{imvrcos\theta} 2\pi r^2 sin\theta\, dr\, d\theta\right|^2$$

$$P = \frac{1}{\left(1 + \left(\frac{mva_0}{2}\right)^2\right)^4} \tag{2}$$

where m is taken to be the mass of the proton.