1 January 2002, Mechanics, Problem 3

1.1 (a)

Parts (a) and (b) can be solved in the same way that we solved May 2001, Mechanics 1. In other words, we assume we are stretching the whole cable to length l, which requires a force kl, and this is equivalent to stretching l/L cables of length L to length L, with the same force. Therefore:

$$F = kl = k\frac{l}{L}L = k_L L$$
$$k_L = \frac{kl}{L} \tag{1}$$

1.2 (b)

We found the general solution for the static case to be:

$$s(x) = -\frac{\rho g x^2}{2kl} + Bx + C$$

Apply boundary conditions $s(0) = 0$ and $T(l) = Mg$:

$$s(0) = C = 0$$
$$T(l) = kl\left(\frac{\partial s}{\partial x}\right)_{x=l} = kl\left(-\frac{\rho g}{k} + B\right) = Mg$$
$$B = \frac{Mg}{kl} + \frac{\rho g}{k} = \frac{(M+m)g}{kl}$$
$$s_0 = s(L) = -\frac{mg}{2k} + \frac{(M+m)g}{k} = \frac{g}{k}\left(M + \frac{m}{2}\right) \tag{2}$$

1.3 (c)

We have the equation (derived in the problem mentioned in part (a)):

$$\rho\ddot{s} = \rho g + kls''$$

The boundary conditions are $s(0) = 0$ and that $s(l)$ matches the equation of motion for the point mass that is hanging. Namely:

$$Mg - T(l) = Mg - kl\left(\frac{\partial s}{\partial x}\right)_{x=l} = M\ddot{s(l)}$$

Here the smart thing to do (which I saw in the Russians' solution) is to define:

$$D(x,t) = s(x,t) - s_0(x)$$

where $s_0(x)$ is the displacement at point x as found in part (b) (for the static case). This way D represents the displacement from the static case, not from the natural length of the string. The equations reduce to:

$$\rho \ddot{D}(x,t) = klD''(x,t)$$
$$M\ddot{D}(l,t) = -klD'(l,t)$$

with boundary condition $D(0,t) = 0$. So D satisfies a wave equation and this boundary condition, which suggests the solution:

$$D(x,t) = e^{i\omega t} sin(px)$$

Plugging this into the first equation gives $p = \frac{\omega}{l}\sqrt{\frac{m}{k}}$, and plugging into the second equation gives:

$$tan\left(\omega\sqrt{\frac{m}{k}}\right) = \frac{\sqrt{km}}{M\omega} \tag{3}$$

This defines the frequency in general.

1.4 (d)

(i)M=0

$$tan\left(\omega\sqrt{\frac{m}{k}}\right) = \infty$$
$$\omega\sqrt{\frac{m}{k}} = (2n-1)\frac{\pi}{2} \quad n\epsilon\mathbb{N}$$
$$\omega_{min} = \sqrt{\frac{k}{m}}\frac{\pi}{2} \tag{4}$$

(ii) m=0

This is the case of a hanging mass on a massless spring, and the frequency is:

$$\omega = \sqrt{k/M} \tag{5}$$

(iii) $m << M$

$$tan\left(\omega\sqrt{\frac{m}{k}}\right) << 1$$
$$tan\left(\omega\sqrt{\frac{m}{k}}\right) \approx \omega\sqrt{\frac{m}{k}} = \frac{\sqrt{km}}{M\omega}$$
$$\omega = \sqrt{\frac{k}{M}} \tag{6}$$