Problem

A space probe is launched from Earth into a transfer orbit whose maximum radius $b$ is slightly larger than the distance from the Sun to Mars. The launch time is such that when the probe reaches distance $b$ from the Sun it has a near collision with Mars, which deflects the velocity of the probe by $\approx 180^\circ$ with respect to Mars and gives a forward boost to the velocity of the probe with respect to the Sun.

What is the largest distance from the Sun to which the probe can now travel?

As an intermediate step, calculate such parameters of the transfer orbit as its eccentricity $e$, characteristic radius $r_c$, energy $E$, angular momentum $L$, and the maximum and minimum velocities $v_a$ and $v_b$.

You may make the approximations that the orbits of Earth and Mars are circular with radii $a$ and $b$, respectively, that the masses of Earth and Mars do not affect the transfer orbit between the two planets, that the mass of the Earth and Sun can be ignored during the near collision between the probe and Mars, and that the masses of Earth and Mars can again be ignored after the near collision. You may also ignore the complication that the distance of closest approach needed for Mars to deflect the probe by $180^\circ$ is less than its radius.

First compute the intermediate parameters $e, r_c, E, L$.

Max $b$ min velocities $v_a \leq v_b$.

Transfer orbit is elliptical $\Rightarrow$ $e$ between $0 \leq 1$.

Radius of orbit governed by $r(\phi) = \frac{c}{1 + e \cos \phi}$.

We have two points $\phi = 0$; $\phi = \pi$.

$r(0) = a \Rightarrow a = \frac{c}{1 + e}$

$r(\pi) = b \Rightarrow b = \frac{c}{1 - e}$

$a(1 + e) = b(1 - e)$

$a + ae = b - be$

$b - a = e(a + b)$

$e = \frac{b - a}{a + b}$

As a check, if $a = b$

the orbit is a circle $\Rightarrow$

$e = 0 \checkmark$.
characteristic radius \( r_0 \rightarrow c \) in our previous calculation
\[ a(1+e) = c \]
\[ \Rightarrow r_0 = c = a \left(1 + \frac{b-a}{a+b}\right) \]
\[ r_0 = a \left(\frac{a+b-b-a}{a+b}\right) = a \left(\frac{2b}{a+b}\right) = \frac{2ab}{a+b} \]
energy \( E \):
get this by energy conservation
\[ E = \frac{1}{2} m v^2 - \frac{GMm}{r} \]
\[ \Rightarrow \text{solving for } v \text{ in terms of } E \]
\[ \frac{1}{m} \left(E + \frac{GMm}{r}\right) = v^2 \Rightarrow v = \sqrt{\frac{2E}{m} + \frac{2GM}{r}} \]
conservation of angular momentum
\[ m v_a a = m v_b b \Rightarrow v_a a = v_b b \]
\[ a \sqrt{\frac{2E}{m} + \frac{2GM}{a}} = b \sqrt{\frac{2E}{m} + \frac{2GM}{b}} \]
\[ a^2 \left(\frac{2E}{m} + \frac{2GM}{a}\right) = b^2 \left(\frac{2E}{m} + \frac{2GM}{b}\right) \]
\[ \frac{2Ea^2}{m} + 2GMA = \frac{2Eb^2}{m} + 2GMB \]
\[ \frac{2E}{m} \left(a^2-b^2\right) = 2GM \left(b-a\right) \]
\[ \Rightarrow E = GMm \left[\frac{b-a}{a^2-b^2}\right] \]
\[ \Rightarrow L = m v_a a \]
\[ L = ma \sqrt{\frac{2E}{m} + \frac{2GM}{a}} = ma \sqrt{\frac{2GM}{m} \left(\frac{b-a}{a^2-b^2}\right) + \frac{2GM}{a}} \]
\[ L = ma \sqrt{2GM \left(\frac{b-a}{a^2b^2+1}\right)} \]
\[ \frac{b-a}{(a+b)(a-b)} + \frac{1}{a} = \frac{-1}{a+b} + \frac{1}{a} = \frac{-a+atb}{a(a+b)} \]
\[ = \frac{b}{a} \left(\frac{1}{a+b}\right) \]
\[ L = ma \sqrt{2GM \left( \frac{b}{a+a+b} \right)} = \sqrt{2GMm^2 \left( \frac{ab}{a+b} \right)} \]

Recognize that \( V_a \) is the square root above

\[ V_a = \sqrt{2GM \left( \frac{b}{a+b} \right)} \]

\[ V_b = \sqrt{2GM \left( \frac{a}{b+a} \right)} \]

\[ L \Rightarrow \frac{b-a}{a^2-b^2} + \frac{1}{b} \]

If we can find the velocity of the probe after the near collision we can find \( r_{max} \) via energy conservation:

In general, total energy for an orbit

\[ T + V = \frac{1}{2} m (r^2 + r^2 \dot{\phi}^2) - \frac{GMM}{r} \]

where \( L = mr^2 \dot{\phi} \)

\[ \Rightarrow \dot{\phi} = \frac{L}{mr^2} \quad \Rightarrow E_{tot} = \frac{1}{2} m (r^2 + r^2 \left( \frac{L^2}{m^2 r^4} \right)) - \frac{GMM}{r} \]

\[ E_{tot} = \frac{1}{2} mr^2 + \frac{L^2}{2mr^2} - \frac{GMM}{r} \]

Given \( L = mv_f b \) after the near collision

if \( E_i = E_f \)

\[ \frac{m^2 v_{f_1} b^2}{2mb^2} - \frac{GMM}{b} = \frac{1}{2} m r_{max}^2 \dot{\phi}^2 - \frac{GMM}{r_{max}} \]

\[ L_f = mv_f b = mr_{max}^2 \dot{\phi} \]

\[ \frac{v_f}{r_{max}} = \dot{\phi} \]

\[ \frac{1}{2} mv_f^2 - \frac{GMM}{b} = \frac{1}{2} m r_{max}^2 \left( \frac{v_f^2 b^2}{r_{max}^2} \right) - \frac{GMM}{r_{max}} \]

\[ \frac{1}{2} mv_f^2 - \frac{GMM}{b} = \frac{1}{2} m v_f^2 \left( \frac{b^2}{r_{max}^2} \right) - \frac{GMM}{r_{max}} \]

We need to determine \( v_f \) then
Considering from Mars frame of reference if Mars is at rest, probe makes a u-turn \( \Rightarrow \) \( x \) component of \( v_b \) switches sign while \( y \) component remains the same.

\[ v_i = v_1 + v_m \]
\[ v_f = -v_2 + v_m \] \( \Rightarrow \) reversal by 180°

\[ v_i = -v_f \Rightarrow v_1 + v_m = v_2 - v_m \]
\[ \Rightarrow v_2 = v_1 + 2v_m \]

\( v_2 = v_f \) in our previous notation

\[ v_f = v_b + 2v_m \]

\[ \frac{1}{2}m(v_b + 2v_m)^2 = \frac{GMm}{b} = \frac{1}{2}m(v_b + 2v_m)^2 \left( \frac{b^2}{r_{max}^2} \right) - \frac{GMm}{r_{max}} \]

\( v_m \) given by \( \text{F=ma} \) eq.

\[ \frac{GMm}{b^2} = \frac{mv_m^2}{b} \]

since Mars orbits in a circle \( r_0 \) radius

\[ v_m = \sqrt{\frac{GM}{b}} \]

\[ v_b + 2v_m = \sqrt{2GM \left( \frac{a}{b(a+b)} \right)} + 2 \sqrt{\frac{GM}{b}} \]

\[ (v_b + 2v_m)^2 = 2GM \frac{a}{b(a+b)} + 4 \left( \frac{GM}{b} \right) \]

\[ + 4 \sqrt{\frac{GM}{b}} \sqrt{2GM \left( \frac{a}{b(a+b)} \right)} \]

\[ = \frac{2GMa}{b(a+b)} + \frac{4GM}{b} + 4GM \sqrt{\frac{2a}{b^2(a+b)}} \]

the rest is algebra \( \Rightarrow \) a quadratic equation to solve for \( r_{max} \) in terms of \( G, M, a, \frac{1}{b} \), plugging in \( v_b + 2v_m \) to starred (*) equation above.