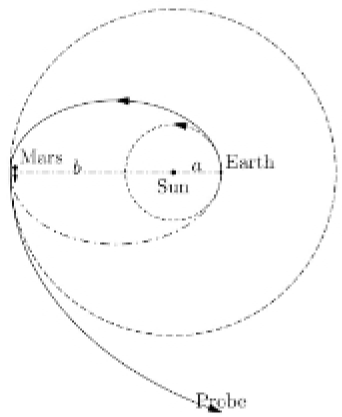
J02M.2—Slingshot Orbit

Problem

A space probe is launched from Earth into a transfer orbit whose maximum radius b is slightly larger than the distance from the Sun to Mars. The launch time is such that when the probe reaches distance b from the Sun it has a near collision with Mars, which deflects the velocity of the probe by $\approx 180°$ with respect to Mars and gives a forward boost to the velocity of the probe with respect to the Sun.



What is the largest distance from the Sun to which the probe can now travel?

As an intermediate step, calculate such parameters of the transfer orbit as its eccentricity ϵ, characteristic radius r_0, energy E, angular momentum L, and the maximum and minimum velocities v_a and v_b.

You may make the approximations that the orbits of Earth and Mars are circular with radii a and b, respectively, that the masses of Earth and Mars do not affect the transfer orbit between the two planets, that the mass of the Earth and Sun can be ignored during the near collision between the probe and Mars, and that the masses of Earth and Mars can again be ignored after the near collision. You may also ignore the complication that the distance of closest approach needed for Mars to deflect the probe by 180° is less than its radius.

First compute the intermediate parameters ε, r_0, E, L, & max & min velocities v_a & v_b

transfer orbit is elliptical $\Rightarrow$ ε between 0 & 1

radius of orbit governed by $r(\phi) = \dfrac{c}{1+\varepsilon\cos\phi}$

we have two points @ $\phi = 0$ & $\phi = \pi$

$$r(0) = a \Rightarrow a = \frac{c}{1+\varepsilon} \qquad a(1+\varepsilon) = c$$

$$r(\pi) = b \Rightarrow b = \frac{c}{1-\varepsilon} \qquad b(1-\varepsilon) = c$$

$$\Rightarrow a(1+\varepsilon) = b(1-\varepsilon)$$

$$a + a\varepsilon = b - b\varepsilon$$

$$b - a = \varepsilon(a+b)$$

$$\varepsilon = \frac{b-a}{a+b}$$

As a check, if $a = b$ the orbit is a circle & $\varepsilon = 0$ ✓

characteristic radius r_o $\rightarrow$ c in our previous calculation

$$a(1+\varepsilon) = c$$

$$\Rightarrow r_o = c = a\left(1 + \frac{b-a}{a+b}\right)$$

$$r_o = a\left(\frac{a+b+b-a}{a+b}\right) = a\left(\frac{2b}{a+b}\right) = \frac{2ab}{a+b}$$

energy E:

get this by energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$\Rightarrow$ solving for v in terms of E

$$\frac{2}{m}\left(E + \frac{GMm}{r}\right) = v^2 \quad \Rightarrow \quad v = \sqrt{\frac{2E}{m} + \frac{2GM}{r}}$$

conservation of angular momentum

$$\Rightarrow mv_a a = mv_b b \quad \Rightarrow \quad v_a a = v_b b$$

$$a\sqrt{\frac{2E}{m} + \frac{2GM}{a}} = b\sqrt{\frac{2E}{m} + \frac{2GM}{b}}$$

$$a^2\left(\frac{2E}{m} + \frac{2GM}{a}\right) = b^2\left(\frac{2E}{m} + \frac{2GM}{b}\right)$$

$$\frac{2Ea^2}{m} + 2GMa = \frac{2Eb^2}{m} + 2GMb$$

$$\frac{2E}{m}(a^2-b^2) = 2GM(b-a)$$

$$E = GMm\left[\frac{b-a}{a^2-b^2}\right]$$

$$\Rightarrow L = mv_a a$$

$$L = ma\sqrt{\frac{2E}{m} + \frac{2GM}{a}} = ma\sqrt{\frac{2GMm}{m}\left(\frac{b-a}{a^2-b^2}\right) + \frac{2GM}{a}}$$

$$L = ma\sqrt{2GM\underbrace{\left(\frac{b-a}{a^2+b^2} + \frac{1}{a}\right)}}$$

$$\frac{b-a}{(a+b)(a-b)} + \frac{1}{a} = \frac{-1}{a+b} + \frac{1}{a} = \frac{-a+a+b}{a(a+b)}$$

$$= \frac{b}{a}\left(\frac{1}{a+b}\right)$$

$$L = ma\sqrt{2GM\left(\frac{b}{a(a+b)}\right)} = \sqrt{2GMm^2\left(\frac{ab}{a+b}\right)}$$

Recognize that v_a is the square root above

$$v_a = \sqrt{2GM\left(\frac{b}{a(a+b)}\right)}$$

$$v_b = \sqrt{2GM\left(\frac{a}{b(a+b)}\right)}$$

$\hookrightarrow$ from $\frac{b-a}{a^2-b^2} + \frac{1}{b}$

If we can find the velocity of the probe after the near collision we can find r_{max} via energy conservation:

in general, total energy for an orbit

$$T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r}$$

where $L = mr^2\dot{\phi}$

$$\Rightarrow \dot{\phi} = \frac{L}{mr^2} \nearrow E_{tot} = \frac{1}{2}m\left(\dot{r}^2 + r^2\left(\frac{L^2}{m^2r^4}\right)\right) - \frac{GMm}{r}$$

$$E_{tot} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

given $L = mv_f b$ after the near collision

if $E_i = E_f$

$$\frac{m^2v_f^2b^2}{2mb^2} - \frac{GMm}{b} = \frac{1}{2}mr_{max}^2\dot{\phi}^2 - \frac{GMm}{r_{max}}$$

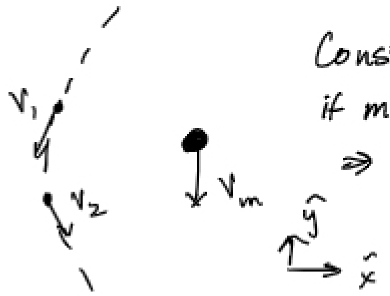
$$L_f = mv_f b = mr_{max}^2\dot{\phi}$$

$$\frac{v_f b}{r_{max}^2} = \dot{\phi}$$

$$\frac{1}{2}mv_f^2 - \frac{GMm}{b} = \frac{1}{2}mr_{max}^2\left(\frac{v_f^2b^2}{r_{max}^4}\right) - \frac{GMm}{r_{max}}$$

$$\frac{1}{2}mv_f^2 - \frac{GMm}{b} = \frac{1}{2}mv_f^2\left(\frac{b^2}{r_{max}^2}\right) - \frac{GMm}{r_{max}}$$

We need to determine v_f then



Considering from mars frame of reference
if mars is at rest, probe makes a u-turn
$\Rightarrow$ x component of v_b switches sign while y component remains the same.
In stationary frame, add v_m back onto velocities

thus $v_i = v_1 + v_m$
$v_f = -v_2 + v_m$ $\leftarrow$ reversal by 180^0

$v_i = -v_f \Rightarrow v_1 + v_m = v_2 - v_m$
$\Rightarrow v_2 = v_1 + 2v_m$

$v_2 = v_f$ in our previous notation

$\Rightarrow v_f = v_b + 2v_m$
↳ velocity of mars
↳ velocity of probe (calculated earlier)

$$\Rightarrow^{*} \frac{1}{2}m(v_b+2v_m)^2 - \frac{GMm}{b} = \frac{1}{2}m(v_b+2v_m)^2\left(\frac{b^2}{r_{max}^2}\right) - \frac{GMm}{r_{max}}$$

v_m given by F=ma eq.

$$\frac{GMm}{b^2} = \frac{mv_m^2}{b}$$ since Mars orbits in a circle w/ radius b

$$v_m = \sqrt{\frac{GM}{b}}$$

$$v_b + 2v_m = \sqrt{2GM\left(\frac{a}{b(a+b)}\right)} + 2\sqrt{\frac{GM}{b}}$$

$$\Rightarrow (v_b+2v_m)^2 = 2GM\frac{a}{b(a+b)} + 4\left(\frac{GM}{b}\right) + 4\sqrt{\frac{GM}{b}}\sqrt{2GM\left(\frac{a}{b(a+b)}\right)}$$

$$= \frac{2GMa}{b(a+b)} + \frac{4GM}{b} + 4GM\sqrt{\frac{2a}{b^2(a+b)}}$$

the rest is algebra $\rightarrow$ a quadratic equation to solve for r_{max} in terms of G, M, a, & b; plugging in $v_b + 2v_m$ to starred (*) equation above