1 January 2002, Mechanics, Problem 2

$$\epsilon = 1 - \frac{2}{\frac{b}{a}+1} = \frac{b/a-1}{b/a+1} \quad Eccentricity \tag{1}$$

I don't know what they mean by characteristic radius.

$$\frac{GMm}{a^2} = \frac{mv_a^2}{a}$$

$$E = \frac{1}{2}mv_a^2 - \frac{GMm}{a} = -\frac{mag}{2} \tag{2}$$

$$L = am\sqrt{ag} \tag{3}$$

$$v_a = \sqrt{ag} \tag{4}$$

$$v_b = \frac{a\sqrt{ag}}{b} \tag{5}$$

As the satellite passes by Mars, it seems that Mars gives it a boost equal to its own velocity, while changing the sign of the original velocity. Thus, the final velocity of the satellite is:

$$u = V_{Mars} - v_b$$

$$\frac{GMM_{Mars}}{b^2} = \frac{M_{Mars}V_{Mars}^2}{b} \rightarrow V_{Mars} = \sqrt{\frac{GM}{b}}$$

Now we plug this into the conservation of energy and angular momentum equation:

$$\frac{1}{2}mu^2 - \frac{GMm}{b} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r}$$

$$mbu = mr^2\dot{\phi}$$

Since they want some maximum distance, I assume that it's not going to be infinity. Therefore, the maximum distance will occur at a point with $\dot{r} = 0$. Plug that in to get:

$$\frac{1}{2}m(V_{Mars} - v_b)^2 - \frac{GMm}{b} = \frac{1}{2}mr_{max}^2\left(\frac{bu}{r_{max}^2}\right)^2 - \frac{GMm}{r_{max}}$$

Solve for r_{max} to obtain:

$$r_{max} = ga^2\frac{\sqrt{1+(1-\sqrt{a/b})^4} - 1}{u^2} \tag{6}$$