Jan 2002 #1

$$V(r) = \begin{cases} 0 & 0 \le r < a \\ -U_0 & a < r < b \\ 0 & b < r \end{cases} \qquad U_0 > 0$$

condition for no bound states?

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

V spherically symmetric: separate the radial and angular dependence

$$\psi = R(r)\,Y_\ell^m(\theta,\phi)$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]R(r)Y_\ell^m(\theta,\phi) + V(r)R(r)Y_\ell^m(\theta,\phi)$$

$$= E\,R(r)\,Y_\ell^m(\theta,\phi)$$

Radial equation:

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{\ell(\ell+1)}{r^2}R\right] + V(r)R = ER$$

The ground state would be for $\ell=0$, so we check solutions for this case:

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + V(r)R = ER \qquad \text{with } V(r) \text{ given above}$$

Then let $U(r) \equiv r R(r)$ or $R(r) = \frac{U(r)}{r}$

$$\Rightarrow \frac{dR}{dr} = \frac{dU}{dr}\frac{1}{r} - \frac{1}{r^2}U(r)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r\frac{dU}{dr} - U\right) = \frac{1}{r^2}\left[\frac{dU}{dr} + \frac{r\,d^2U}{dr^2} - \frac{dU}{dr}\right] = \frac{1}{r}\frac{d^2U}{dr^2}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2U}{dr^2} - (E - V(r))U = 0$$

$$\frac{d^2U}{dr^2} + \frac{2m}{\hbar^2}(E - V(r))U = 0$$

region I: $r < a$ II: $a < r < b$ III: $b < r$

in I and III: $\frac{d^2U}{dr^2} + \frac{2mE}{\hbar^2}U = 0 \Rightarrow K = \sqrt{\frac{2m(-E)}{\hbar^2}}$ $\quad U \sim e^{\pm Kr}$

region I: $U = C_1 e^{\kappa r} + C_2 e^{-\kappa r}$ $\quad U(r\to 0)=0 \Rightarrow U = \sinh(\kappa r)$

III: $U = C_3 e^{-\kappa(r-b)}$

region II: $\frac{d^2U}{dr^2} = \frac{2m}{\hbar^2}\left(-U_0 - E\right)U = -q^2 U$ $\quad q^2 = \frac{2m}{\hbar^2}\left(E+U_0\right) > 0$

(if $E < -U_0$ no bound states, so $E > -U_0$)

$U = C_1 \sin q(r-a) + C_2 \cos q(r-a)$

different ℓ than before $\downarrow$

U continuous at a: $\sinh \kappa a = C_2$

" $\quad b$: $C_1 \sin q\ell + C_2 \cos q\ell = C_3$ $\quad \ell \equiv b-a$

U' continuous at a: $\kappa \cosh \kappa a = q C_1$

U' " $\quad b$: $q\left(C_1 \cos q\ell - C_2 \sin q\ell\right) = -\kappa C_3$

Now, assume $E \to 0$ ("least bound" states; can also proceed without assuming this but the constraint equation is ridiculous)

then $\kappa \to 0$, $q^2 \to \frac{2mU_0}{\hbar^2}$. Solve to order κ:

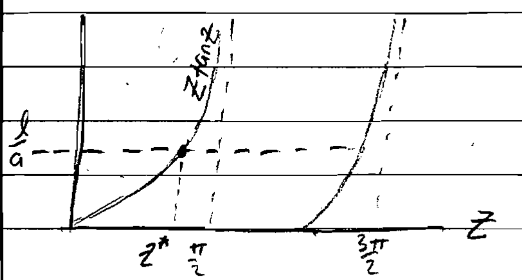
$C_2 = \kappa a$, $C_1 = \frac{\kappa}{q}$: $\quad \frac{\kappa}{q}\sin q\ell + \kappa a \cos q\ell = C_3$

$\kappa \cos q\ell - \kappa q a \sin q\ell = -\kappa C_3 \Rightarrow \cos q\ell - q a \sin q\ell = -C_3$

but C_3 is $\mathcal{O}(\kappa)$, so the constraint is

$\cos q\ell = q a \sin q\ell$: $\quad$ let $z \equiv q\ell$

Then $z \tan z = \frac{\ell}{a} = \frac{b-a}{a}$ is the constraint



z^* is the smallest solution to $z\tan z = \frac{\ell}{a}$

$z^* < \frac{\pi}{2}$

For no solution, require $z < z^*$

$q^2\ell^2 < z^{*2}$

exact: $\Longrightarrow \frac{2mU_0}{\hbar^2} < \frac{z^{*2}}{(b-a)^2}$ where z^* is the smallest positive solution to $z\tan z = \frac{\ell}{a}$