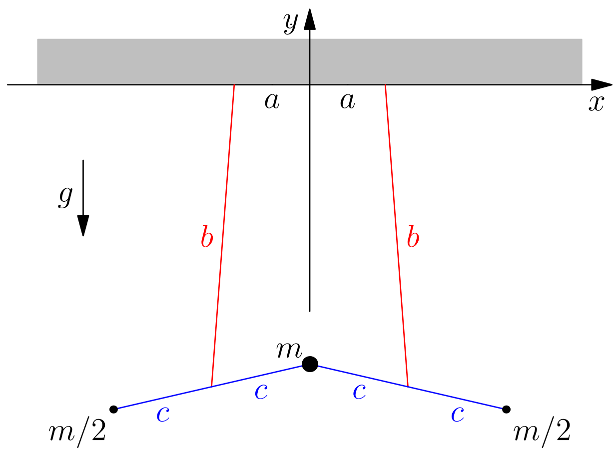
J02M.1—Flapping Toy

Problem

Deduce the frequency of small oscillations of the flapping toy shown in the figure below, supposing the central mass m moves only vertically, and the motion of the others masses is only in the x-y plane.

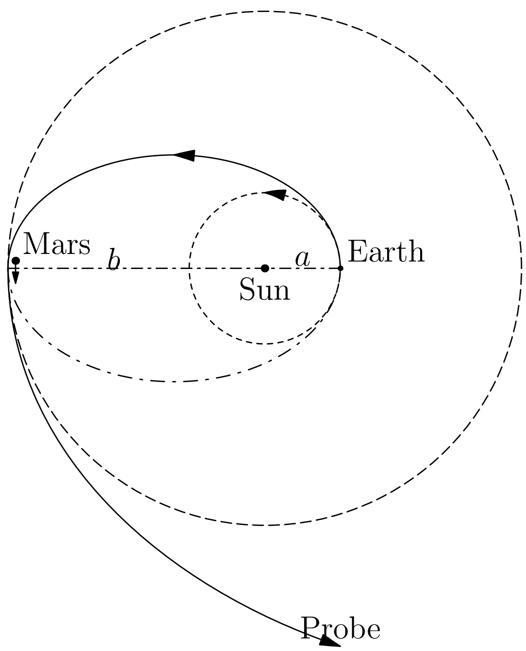


The flapping toy consists of the central mass m connected by two massless rods of length $2c$ to two masses $m/2$. The centers of the two rods are suspended from a horizontal plane by massless strings of length b, with distance $2a$ between the upper points of suspension. Length c is slightly larger than a.

J02M.2—Slingshot Orbit

Problem

A space probe is launched from Earth into a transfer orbit whose maximum radius b is slightly larger than the distance from the Sun to Mars. The launch time is such that when the probe reaches distance b from the Sun it has a near collision with Mars, which deflects the velocity of the probe by $\approx 180°$ with respect to Mars and gives a forward boost to the velocity of the probe with respect to the Sun.



What is the largest distance from the Sun to which the probe can now travel?

As an intermediate step, calculate such parameters of the transfer orbit as its eccentricity ϵ, characteristic radius r_0, energy E, angular momentum L, and the maximum and minimum velocities v_a and v_b.

You may make the approximations that the orbits of Earth and Mars are circular with radii a and b, respectively, that the masses of Earth and Mars do not affect the transfer orbit between the two planets, that the mass of the Earth and Sun can be ignored during the near collision between the probe and Mars, and that the masses of Earth and Mars can again be ignored after the near collision. You may also ignore the complication that the distance of closest approach needed for Mars to deflect the probe by 180° is less than its radius.

This problem is an example of a 4-body gravitational interaction. Amusing web sites on the n-body problem are `http://www.soe.ucsc.edu/~charlie/3body/` and
`http://www.ams.org/new-in-math/cover/orbits1.html`

J02M.3—Stretching Cable

Problem

Consider a cable of mass m with its upper end fixed and with mass M suspended from its lower end. When the cable is unstretched (and NOT hanging vertically) it has length l and spring constant k.

a) What is the spring constant of a segment of the cable with length $L \ll l$? You may assume that the cable has a fixed cross section, and is made from a material with a constant Young modulus.

b) Determine the amount s_0 by which the cable is stretched in the static situation when the suspended mass M is at rest. Make sure to take in to account the effect of the mass of the cable.

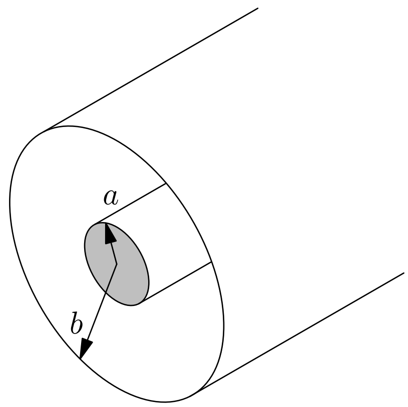
c) Deduce an expression for the possible angular frequencies of vertical oscillations of the mass M.

d) What is the lowest frequency of oscillation for the special cases that (i) $M = 0$, (ii) $m = 0$, and (iii) $m/M \ll 1$?

J02E.1—Coaxial Transmission Line

Problem

An infinitely long coaxial transmission line made from perfect conductors lies along the z axis, as shown below. The inner wire is of radius a, the outer wire is a cylinder of inner radius b, and the space between is filled with a material of (relative) dielectric constant ϵ and (relative) permeability μ.

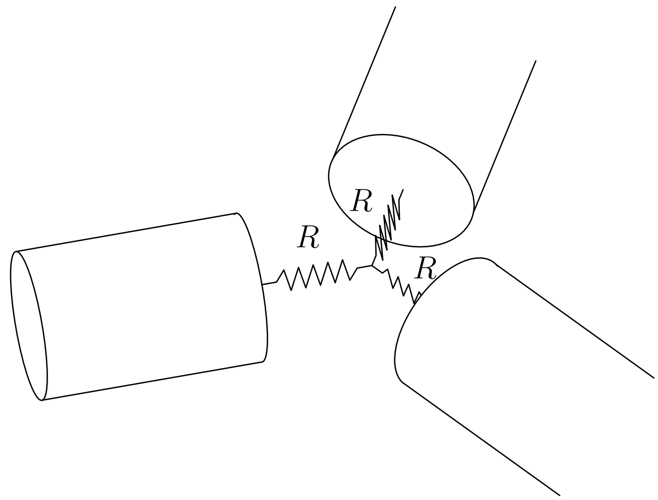


a) Find the speed v of the waves down the transmission line, the magnitude of the ratio E/B of the electric and magnetic fields, and the impedance $Z = V(z,t)/I(z,t)$ of the transmission line where $I(z,t)$ is the current in each of the wires and $V(z,t)$ is the voltage difference between the two wires.

As on all parts of this exam, either MKSA or Gaussian units may be employed.

b) A transmission line of impedance Z_1 for $z < 0$ is connected to a line of impedance Z_2 for $z > 0$. A wave $E_0 e^{i(kz-\omega t)}$ is incident from $z = -\infty$. Derive an expression for the amplitudes of the transmitted and reflected waves.

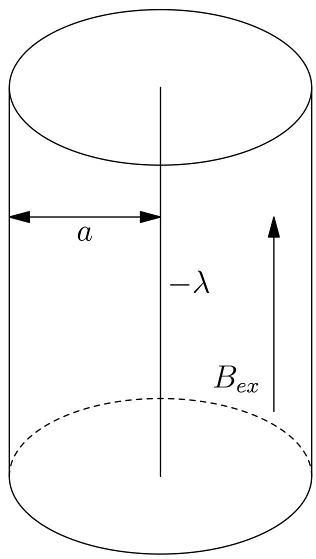
c) Assume the answer to part a) for the impedance of the transmission line was Z. It is desired to split the signal from the transmission line into two identical lines as shown in the Figure. What value of R for the matching resistors will ensure that there are no reflections?


R
R
R

J02E.2—Rotating Dielectric Cylinder

Problem

An infinitely long wire with linear charge density $-\lambda$ lies along the z axis. An insulating cylindrical shell of radius a and moment of inertia I per unit length is concentric with the wire, and can rotate freely about the z axis. The areal charge density on the cylinder is $\sigma = \lambda/2\pi a$ and is uniformly distributed.



The cylinder is immersed in an external magnetic field $B_{\mathrm{ex}}\hat{\mathbf{z}}$, and is initially at rest.

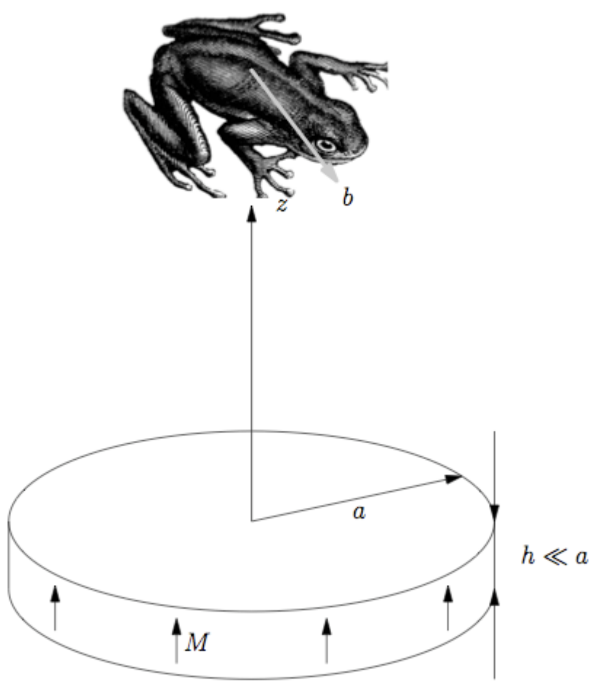
Starting at $t = 0$ the external magnetic field is slowly reduced to zero over a time $T \gg a/c$, where c is the speed of light. What is the final angular velocity ω of the cylinder?

J02E.3—Levitating Frog

Problem

It has proven possible to levitate objects (frogs!) on the surface of the earth in regions of high magnetic field gradients. This problem explores how a "spherical frog" might be levitated above a permanent magnet.

a) Consider a magnetic disk of radius a and thickness $h \ll a$. The magnetic material has a constant magnetic moment/volume $\mathbf{m}$ oriented parallel to the axis of the disk, the z axis. Find the magnetic field $\mathbf{B}(z)$ along the z axis.



b) The "spherical frog" to be levitated has a radius b and mass k and (relative) diamagnetic permeability μ. Assume that $b \ll a$, so that the magnetic field is roughly constant across the frog. Find the maximum value for the mass k for there to be an equilibrium point above the disk in terms of m, a, b, h, μ, and the position z_0 above the disk where that occurs.

Note: the magnetic moment $\mathbf{M}$ induced in a solid diamagnetic sphere of (relative) permeability μ and volume V by an external magnetic field $\mathbf{B}$ is given by $\mathbf{M} = (\mu - 1)\mathbf{B}V/4\pi$ in Gaussian units (and $\mathbf{M} = (\mu - 1)\mathbf{B}V$ in MKSA units).

After this exam is over you may wish to show that the equilibrium point is stable against small axial and radial perturbations provided $a/\sqrt{7} < z_0 < \sqrt{2/5}a$.

J02Q.1—Particle in a Spherically Symmetric Potential

Problem

A particle of mass m moves in the spherically symmetrical potential in 3 dimensions:

$$V(r) = \begin{cases} 0, & 0 \le r < a, \\ -U_0, & a < r < b, \\ 0, & b < r \end{cases}$$

where $U_0 > 0$.

What is the condition on U_0 so that there will not be any bound states?

J02Q.2—Wave Functions and Coordinate Frames

Problem

In this problem you will study the way in which a wave function changes from one coordinate frame to another and then apply it to a physical problem.

a) A particle of mass m is described by a wave function $\psi(x,t)$ in the lab frame. What would be the wave function that describes the particle for an observer that moves with velocity v in the positive x direction?

   (**Hint:** One way to find the answer is by decomposing $\psi(x,t)$ into plane waves.)

b) A hydrogen atom is at rest when a neutron collides with the nucleus and causes it to move with velocity v. What is the probability that the atom will remain in its ground state after the collision?

   Since the proton is much heavier than the electron, you may neglect corrections of the order m_e/m_p (the ratio of the electron to proton mass). You may also assume that the collision between the proton and neutron is instantaneous.

J02Q.3—Interacting Particles on a Line

Problem

Two identical neutral, spin-$\frac{1}{2}$ particles of mass M and magnetic moment μ are restricted to move on a line. The interaction between them is spin dependent and is described by the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2m} + (2\hbar^2 - S_T^2)U_0(x_1 - x_2).$$

Here $\mathbf{S}_T = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin of the system ($S_T = 0$ or 1) and U_0 is an infinite-well potential:

$$U_0(x) = \begin{cases} -\frac{\pi^2}{4ma^2}, & |x| < a, \\ \infty, & a < |x|. \end{cases}$$

a) Find the energy eigenstates of the system (in zero magnetic field) and their corresponding wave-functions. What is the energy E_0 of the ground state?

b) Assume that initially the system is in the ground state and that $\hbar\omega + E_0 > 0$. To first order in perturbation theory, what is the half lifetime of the bound state in the presence of an electromagnetic plane wave with magnetic field $B_z = B_0 \cos k(x - ct)$.

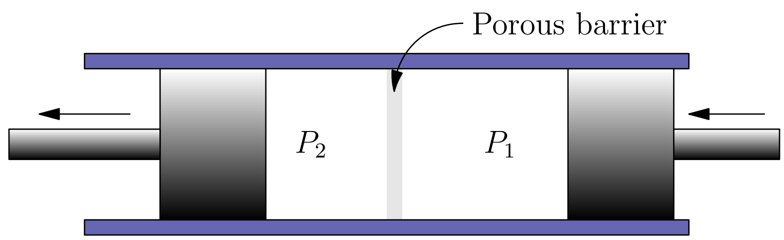
   You may use the nonrelativistic approximation $ka \ll 1$ and expand to lowest order in ka.

J02T.1—Joule-Thomson Process

Problem

A thermally isolated vessel containing a non-ideal gas is separated in two parts by a porous barrier. Initially all of the gas is on one side of the barrier and occupies a volume V. The gas is transferred slowly through the barrier by moving two pistons inward and outward, while keeping the pressures P_1 and P_2 fixed on both sides of the barrier. This is called a Joule-Thomson process. For an ideal gas the temperatures T_1 and T_2 before and after the process are the same. For a non-ideal gas there will be a small difference $\Delta T = T_2 - T_1$. The problem is to determine ΔT for a non-ideal gas described by the van der Waals equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT.$$



In this problem we assume that the pressure difference is small, so that after the process the volume has increased only by a small amount $\Delta V = V_2 - V_1$.

a) Calculate the free energy $F(V, T)$ for a van der Waals gas with total specific heat C_V.

b) Show that the enthalpy $H \equiv U + PV$ is constant for a Joule-Thomson process.

Hint: Argue that

$$\Delta T = \left(\frac{\partial T}{\partial V}\right)_H \Delta V.$$

c) Find the enthalpy H for a van der Waals gas as a function of V and T.

d) Show that ΔT is positive for high temperature and negative at low temperatures. The temperature T_{inv} at which ΔT changes sign is called the inversion temperature. Derive that

$$T_{\text{inv}} = \frac{2a}{bR}\left(1 - \frac{b}{V}\right)^2.$$

J02T.2—Neutrons in a Magnetic Field

Problem

Consider a degenerate gas of N nonrelativistic neutrons of magnetic moment μ_B in a volume V. The gas is placed in a constant magnetic field H. The problem is to determine the magnetic moment M of the neutron gas, and its susceptibility $\chi = \partial M/\partial H$ at temperature $T = 0$.

a) Derive integral expressions for the average number of neutrons N^+ (N^-) with spin up (down) as a function of the chemical potential μ for $T \neq 0$.

b) Evaluate the integrals in the limit $T \to 0$, where $\mu \to \epsilon_{\mathrm{F}} F$, the Fermi energy.

c) Express the magnetization M in terms of the Fermi energy ϵ_{F}. Find the condition that determines ϵ_{F} in terms of N and $\mu_B H$.

d) Use your result from c) to calculate the susceptibility χ for $\mu_B H \ll \epsilon_F$ at $T = 0$.

J02T.3—Ensemble of Harmonic Oscillators

Problem

Consider an ensemble of $N \gg 1$ independent identical oscillators of natural frequency ω. Suppose there is a total of M quanta (bosons) to distribute among the ensemble. The number of *distinct* ways to do so may be shown to be

$$W(M) = \frac{(M+N-1)!}{M!(N-1)!}\,.$$

a) Write down the internal energy E and the entropy S of the ensemble in terms of M, N and ω.

b) Now suppose the system comes to equilibrium with a heat reservoir at temperature T. By minimizing an appropriate thermodynamic function, find the average distribution $n(T) = \langle M \rangle / N$.

c) Derive the heat capacity C_V versus T.

d) Verify that at equilibrium the derivative $d\langle S \rangle / dE$ gives the inverse temperature.

e) Derive the equation above. [*Hint*: Think partitions.]