Seth Dorfman PHY 511 September 18, 2005

Homework # 1 - Page 3

3) a. ideal gas $\Rightarrow U(T,V,N) = U(T,N)$

$\Rightarrow dU = 0$ during isothermal compression

$dU = 0 = \delta Q - dW$

$\delta Q = dW = P\,dV = \frac{NkT_1}{V}\,dV$

$\Delta Q = \int_{V_1}^{V_2} NkT_1 \frac{dV}{V}$

$\Delta Q = NkT_1 \ln\left(\frac{V_2}{V_1}\right)$

$Q_{yielded} = NkT_1 \ln\left(\frac{V_1}{V_2}\right)$

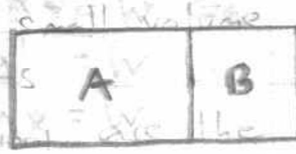
b. Work is done on the large (A) volume by the small (B) one ok

$(\delta Q_{system} = 0) \Rightarrow \delta Q_A + \delta Q_B = 0)$

$V_A + V_B = const = V_1 + V_2 \Rightarrow dV_A + dV_B = 0$

$\delta Q_A = dU_A + dW_A$

$\delta Q_B = dU_B + dW_B$



$dW = dW_A + dW_B$

$dW = P_A dV_A + P_B dV_B$

$dW = (P_B - P_A)\,dV_B$

$dW = \left(\frac{NkT_B}{V_B} - \frac{NkT_A}{V_A}\right) dV_B$

$dW = Nk\left(\frac{T_B}{V_B} - \frac{T_A}{V_1+V_2-V_B}\right) dV_B$ (1)

unfortunately, T_1 is NOT fixed (see solutions)

small volume expands $\Rightarrow T_B$ could dec from T_1 or remain const

large volume contracts $\Rightarrow T_A$ could inc from T_1 or remain const

If T_B decreases as T_A increases, then dW will decrease from equation (1). Therefore in order to maximize the work, we want to allow the temperatures to equilibrate after each infintesimal amount of work done. $\therefore T_A = T_B$ at all points in time. ok

oh oh

Let T_2 = the final temperature of the system,

$\delta Q_{system} = 0 \Rightarrow S_{initial} = S_{final}$

For an ideal gas $S(T,V,N) \sim \ln(TV^{\gamma-1})$

$S_{initial} = S_{final}$ (N is the same for A & B and cancels)

$\Rightarrow \ln(T_1 V_1^{\gamma-1}) + \ln(T_1 V_2^{\gamma-1}) = \ln(T_2 V_{fa}^{\gamma-1}) + \ln(T_2 V_{fb}^{\gamma-1})$

Since $T_{fA} = T_{fB} = T_2$ and $P_{fa} = P_{fb}$ $\Rightarrow$ $V_{fb} = V_{fa} = V_f = \frac{V_1+V_2}{2}$

$$T_1^2 (V_1 V_2)^{\gamma-1} = T_2^2 (V_{fa} V_{fb})^{\gamma-1}$$
$$T_1^2 (V_1 V_2)^{\gamma-1} = T_2^2 (V_f^2)^{\gamma-1}$$
$$T_2 = T_1 \left(\frac{V_1 V_2}{V_f^2}\right)^{(\gamma-1)/2}$$
$$T_2 = T_1 \left(\frac{4V_1 V_2}{(V_1+V_2)^2}\right)^{(\gamma-1)/2}$$

very good!

Since $\delta Q_{system} = 0$, the work done by the system equals the change in internal energy.

$U = \frac{x}{2} N k_B T$ where x is the number of degrees of freedom (for an ideal gas)

$U_{initial} = \frac{x}{2} N k_B T_1 + \frac{x}{2} N k_B T_1$ (N = number of part. in A = number of part. in B)

$U_{final} = \frac{x}{2} N k_B T_2 + \frac{x}{2} N k_B T_2$

$\delta Q_{sys} = 0 = dU_{sys} + dW_{sys}$

For ideal gas

$$\gamma = \frac{C_P}{C_V} = \frac{\frac{x}{2}+1}{x/2}$$
$$\gamma = 1 + \frac{2}{x}$$
$$x = \frac{2}{\gamma - 1}$$

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$\Rightarrow W = -\Delta U_{sys}$
$$W = -[x N k_B (T_2 - T_1)]$$
$$W = x N k_B (T_1 - T_2)$$
$$W = x N k_B T_1 \left[1 - \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(\gamma-1)/2}\right]$$
$$W = x N k_B T_1 \left[1 - \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)}\right]$$

c) $$\frac{dQ}{dV_2} = \frac{\partial}{\partial V_2}\left(N k_B T_1 \ln\left(\frac{V_1}{V_2}\right)\right)$$
$$= N k_B T_1 \left(\frac{-1}{V_2}\right)$$
$$\frac{\partial Q}{\partial V_2} = \frac{-N k_B T_1}{V_2} \quad (2)$$
$$\frac{\partial W}{\partial V_2} = \frac{\partial}{\partial V_2}\left(x N k_B T_1 \left[1 - \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)}\right]\right)$$
$$= -x N k_B T_1 \cdot \frac{1}{x}\left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x - 1)} \cdot \left[\frac{4V_1}{(V_1+V_2)^2} - 2\frac{4V_1V_2}{(V_1+V_2)^3}\right]$$
$$= -N k_B T_1 \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)} \cdot \left[\frac{1}{V_2} - \frac{2}{V_1+V_2}\right]$$
$$= -N k_B T_1 \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)} \left[\frac{V_1+V_2-2V_2}{V_2(V_1+V_2)}\right]$$
$$= \frac{-N k_B T}{V_2} \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)} \left(\frac{V_1-V_2}{V_1+V_2}\right)$$
$$\frac{\partial W}{\partial V_2} = \frac{\partial Q}{\partial V_2} \left(\frac{4V_1V_2}{(V_1+V_2)^2}\right)^{(1/x)} \left(\frac{V_1-V_2}{V_1+V_2}\right)$$

For any V_1 & V_2 $\quad (V_1 - V_2)^2 \geq 0$
$$V_1^2 - 2V_1V_2 + V_2^2 \geq 0$$
$$V_1^2 + 2V_1V_2 + V_2^2 \geq 4V_1V_2$$
$$(V_1+V_2)^2 \geq 4V_1V_2$$

3c (continued)

Also for any V_1, V_2 $\quad V_1 + V_2 \geq V_1 - V_2$ $\quad (V_1, V_2 \geq 0)$

$\therefore \frac{\partial W}{\partial V_2} = \frac{\partial Q}{\partial V_2} (\text{pos} \leq 1)^{(1/x)} (\text{pos} < 1)^3$ $\quad$ ($\frac{1}{x}$ also pos < 1)

$\Rightarrow \left|\frac{\partial W}{\partial V_2}\right| = \left|\frac{\partial Q}{\partial V_2}\right| \Delta$ where $0 \leq \Delta < 1$ $\quad$ (3)

For $V_1 = V_2$ $\quad W = Q = 0$ (from expressions in part (a) and (b))

In the problem $0 \leq V_2 < V_1$

From (2) $\frac{\partial Q}{\partial V_2} < 0$

From (3) $\frac{\partial W}{\partial V_2} > \frac{\partial Q}{\partial V_2}$ (since both are negative)

As we decrease V_2 from V_1, W and Q increase from 0. Since $\frac{\partial Q}{\partial V_2}$ is smaller (i.e. more negative) Q increases faster than W. Thus $Q > W$ for $0 \leq V_2 < V_1$ as is the case in the problem.