3) a. ideal gas \( U(T, V, N) = U(T, N) \)  
\[ dU = 0 \text{ during isothermal compression} \]
\[ dU = \delta Q - dW \]
\[ \delta Q = dW = P_dV = \frac{NKT_1}{V} dV \]
\[ \Delta Q = S V_1 \frac{NKT_1}{V} \]
\[ \Delta Q = NKT_1 \ln \left( \frac{V_2}{V_1} \right) \]
\[ Q_{sys} = NKT_1 \ln \left( \frac{V_2}{V_1} \right) \]

b. Work is done on the large volume by the small one \( \text{OK} \)

\[ (\delta Q_{system} = 0) \Rightarrow \delta Q_A + \delta Q_B = 0 \]
\[ \delta W_A + \delta W_B = 0 \]
\[ \delta Q_A = \delta W_A = - \frac{NKT_1}{V_1} dV_1 \]
\[ \delta Q_B = \delta W_B = \frac{NKT_1}{V_2} dV_2 \]
\[ dW = \delta W_A + \delta W_B \]
\[ dW = P_dV_A + P_B dV_B \]
\[ dW = (P_B - P_A) dV_B \]
\[ dW = \left( \frac{NKT_2}{V_B} - \frac{NKT_A}{V_A} \right) dV_B \]

\[ dW = NKT \left( \frac{T_B}{V_B} - \frac{T_A}{V_A} \right) \]

\[ dW = N K \left( \frac{T_B}{V_B} - \frac{T_A}{V_A} \right) dV_B \]

\[ \text{Unfortunately, } T_B \text{ is fixed} \]

\[ \text{small volume expands } \Rightarrow T_B \text{ could dec from } T_2 \text{ or remain const} \]
\[ \text{large volume contracts } \Rightarrow T_A \text{ could inc from } T_1 \text{ or remain const} \]
\[ \text{If } T_B \text{ decreases as } T_A \text{ increases, then } dW \text{ will decrease from eqn (1). Therefore in order to} \]
\[ \text{maximize the work, we want to allow the temperatures to equilibrate after each infinitesimal amount of} \]
\[ \text{work done. i.e. } T_A = T_B \text{ at all points in time. ok} \]

\[ T_2 = \text{the final temperature of the system,} \]
\[ \delta Q_{system} = 0 \Rightarrow S_{initial} = S_{final} \]

For an ideal gas \( S = \frac{C_V}{T} \ln \left( \frac{T}{V_{ref}^{\gamma-1}} \right) \]

\[ S_{initial} = S_{final} \Rightarrow \left( C_V \text{ is the same for } A \text{ and } B \right) \]
\[ \Rightarrow \ln \left( \frac{T_1 V_1^{\gamma-1}}{V_{ref}^{\gamma-1}} \right) + \ln \left( \frac{T_1 V_2^{\gamma-1}}{V_{ref}^{\gamma-1}} \right) = \ln \left( \frac{T_2 V_B^{\gamma-1}}{V_{ref}^{\gamma-1}} \right) + \ln \left( \frac{T_2 V_B^{\gamma-1}}{V_{ref}^{\gamma-1}} \right) \]
Since \( T_{FB} = T_2 \) and \( P_{FB} = P_f \), \( V_{FB} = V_{fa} = V_f = \frac{V_1 + V_2}{2} \)

\[
T_1^2 (V_1 V_2)^{\gamma - 1} = T_2^2 (V_{fa} V_{FB})^{\gamma - 1} = T_2^2 (V_f^2)^{\gamma - 1}
\]

\[
T_2 = T_1 \left( \frac{V_1 V_2}{V_1 + V_2} \right)^{(\gamma - 1)/2}
\]

\[
T_2 = T_1 \left( \frac{4V_1 V_2}{(V_1 + V_2)^2} \right)^{(\gamma - 1)/2}
\]

Since \( \delta Q_{\text{sys}} = 0 \), the work done by the system equals the change in internal energy.

\[
U = \frac{x}{2} N k_B T \text{ where } x \text{ is the number of degrees of freedom (for an ideal gas)}
\]

\[
U_{\text{initial}} = \frac{x}{2} N k_B T_1 + \frac{x}{2} N k_B T_1 \quad (N = \text{number of particles in A})
\]

\[
U_{\text{final}} = \frac{x}{2} N k_B T_2 + \frac{x}{2} N k_B T_2 \quad (N = \text{number of particles in B})
\]

\[
\delta Q_{\text{sys}} = 0 = \delta U_{\text{sys}} + \delta W_{\text{sys}}
\]

\[
\Rightarrow W = -\Delta U_{\text{sys}}
\]

\[
W = -\left[ xN k_B (T_2 - T_1) \right]^x
\]

\[
W = xN k_B (T_1 - T_2) \quad \Rightarrow x = \frac{2}{\gamma - 1}
\]

\[
W = xN k_B T_1 \left[ 1 - \left( \frac{4V_1 V_2}{(V_1 + V_2)^2} \right)^{(\gamma - 1)/2} \right]
\]

\[
W = xN k_B T_1 \left[ 1 - \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

\[
C_v = \frac{\partial Q}{\partial V_2} = \frac{xN k_B T_1}{V_2} \quad (2)
\]

\[
\frac{\partial Q}{\partial V_2} = -\frac{xN k_B T_1}{V_2}
\]

\[
\frac{\partial W}{\partial V_2} = \frac{xN k_B T_1}{V_2} \left[ 1 - \left( \frac{4V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

\[
= \frac{xN k_B T_1}{V_2} \cdot \left[ \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} - 1 \right]
\]

\[
= \frac{xN k_B T_1}{V_2} \left[ \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} - \frac{1}{x} \cdot \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

\[
= \frac{xN k_B T_1}{V_2} \left[ \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} - \frac{1}{x} \cdot \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

\[
= \frac{xN k_B T_1}{V_2} \left[ \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} - \frac{1}{x} \cdot \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

\[
= \frac{xN k_B T_1}{V_2} \left[ \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} - \frac{1}{x} \cdot \left( \frac{V_1 V_2}{(V_1 + V_2)^2} \right)^{(1/x)} \right]
\]

For any \( V_1, V_2 \), \( (V_1 - V_2)^2 \geq 0 \)

\[
V_1^2 - 2V_1 V_2 + V_2^2 \geq 0
\]

\[
V_1^2 + 2V_1 V_2 + V_2^2 \geq 4V_1 V_2
\]

\[
(V_1 + V_2)^2 \geq 4V_1 V_2
\]
Also for an\( x \) \( V_1 \), \( V_2 \), \( V_1 + V_2 \geq V_1 - V_2 \quad (V_1, V_2 > 0) \)
\[
\frac{\Delta W}{\Delta V_2} = \frac{\Delta Q}{\Delta V_2} (\ast \ast \leq 1) (\# \times) (\ast \ast \leq 1) \quad (\frac{1}{x} \text{ is } \ll 1)
\]
\[
\Rightarrow \frac{\Delta W}{\Delta V_2} = \frac{\Delta Q}{\Delta V_2} \Delta \quad (2) \text{ where } 0 \leq \Delta \leq 1 
\]

For \( V_1 = V_2 \), \( W = Q = 0 \) (from expressions in part (a) and (b))

In the problem, \( 0 \leq V_2 < V_1 \), \( W > 0 \), \( Q > 0 \)

From (2), \( \frac{\Delta Q}{\Delta V_2} < 0 \) and \( \Delta V_2 < V_1 \). \( W \) and \( Q \) increase from 0.

From (3), \( \frac{\Delta W}{\Delta V_2} > \frac{\Delta Q}{\Delta V_2} \) (since both are negative).

As we decrease \( V_2 \) from \( V_1 \), \( W \) and \( Q \) increase from 0.

Since \( \frac{\Delta Q}{\Delta V_2} \) is smaller (i.e., more negative), \( Q \) increases faster than \( W \).

Thus, \( Q > W \) for \( 0 \leq V_2 < V_1 \), as is the case in the problem.