Seth Dorfman Prelims December 19, 2005

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3) a. $H = g\mu_B \vec{B}\cdot\vec{S} = -g\mu_B B S_x = -\frac{1}{2} g\mu_B B \hbar \sigma_x = -\omega_0 \hbar \sigma_x$

$|\psi(t=0)\rangle = |\uparrow\rangle = \binom{1}{0} = \frac{1}{2}\left[\binom{1}{1} + \binom{1}{-1}\right]$ $\quad \omega_0 = \frac{1}{2} g\mu_B B$

$= \frac{1}{\sqrt{2}}\left[|\uparrow_x\rangle + |\downarrow_x\rangle\right]$

$|\psi_t\rangle = \frac{1}{\sqrt{2}}\left[e^{-iEt/\hbar}|\uparrow_x\rangle + e^{-iEt/\hbar}|\downarrow_x\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[e^{i\omega_0 t}|\uparrow_x\rangle + e^{-i\omega_0 t}|\downarrow_x\rangle\right]$

$= \frac{1}{\sqrt{2}}\left[e^{i\omega_0 t}\left[\binom{1}{0} + \binom{0}{-1}\right] + e^{-i\omega_0 t}\left[\binom{1}{0} - \binom{0}{-1}\right]\right]$

$= \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)|\uparrow\rangle + \frac{1}{2}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)|\downarrow\rangle$

$|\psi_t\rangle = \cos(\omega_0 t)|\uparrow\rangle + i\sin(\omega_0 t)|\downarrow\rangle$

$P(\uparrow, \text{one meas}) = \cos^2(\omega_0 \frac{T}{N})$

$P(\uparrow, N \text{ meas}) = \left[\cos^2(\omega_0 \frac{T}{N})\right]^N$ (where $\omega_0 = \frac{1}{2} g\mu_B B$)

For large N $\cos(\omega_0 \frac{T}{N}) \approx 1 - \frac{1}{2}(\omega_0 \frac{T}{N})^2$

$P(\uparrow, N \text{ meas}) \approx \left(1 - \frac{1}{2}(\omega_0 \frac{T}{N})^2\right)^{2N}$

$\approx 1 - 2N \cdot \frac{1}{2}(\omega_0 \frac{T}{N})^2$

$\approx 1 - \frac{1}{N}(\omega_0 T)^2$

If the spin were first measured at $t = T$, $P(\uparrow) = 0$ for the first time when $\cos^2(\omega_0 T) = 0$

$\Rightarrow T = \frac{\pi}{2\omega_0}$, $\omega_0 T = \frac{\pi}{2}$

$\therefore P(\uparrow, N \text{ meas}) = \left(\cos\left(\frac{\pi}{2N}\right)\right)^{2N}$

For large N, this approches one as shown above and

$P(\uparrow, N \text{ meas})_{\text{large } N} \approx 1 - \frac{1}{N}\frac{\pi^2}{4}$

b. $P(\uparrow, 1) = \cos^2(\omega_0 \frac{T}{N})$

$P(\uparrow, 2) = \underbrace{\left(\cos^2(\omega_0 \frac{T}{N})\right)^2}_{\text{remains } |\uparrow\rangle \text{ twice}} + \underbrace{\left(\sin^2(\omega_0 \frac{T}{N})\right)^2}_{\text{or switches twice}}$

$P(\uparrow, 3) = \left(\cos^2(\omega_0 \frac{T}{N})\right)^3 + 3\cos^2(\omega_0 \frac{T}{N})\left(\sin^2(\omega_0 \frac{T}{N})\right)^2$

As seen from the pattern, we are looking for terms in the binomial expansion of $\left(\cos^2(\omega_0 \frac{T}{N}) + \sin^2(\omega_0 \frac{T}{N})\right)^N$ which have even powers of $\sin^2(\omega_0 \frac{T}{N})$. Therefore:

$$P(\uparrow, N) = \frac{1}{2}\{(\cos^2(w_0\tfrac{T}{N}) + \sin^2(w_0\tfrac{T}{N}))^N$$
$$+(\cos^2(w_0\tfrac{T}{N}) - \sin^2(w_0\tfrac{T}{N}))^N\}$$
$$= \frac{1}{2}\{1 + (\cos(2w_0\tfrac{T}{N}))^N\} \qquad w_0T = \frac{\pi}{2}$$
$$= \frac{1}{2}[1 + (\cos(\tfrac{\pi}{N}))^N]$$
$$P(\uparrow, N) = \frac{1}{2} + \frac{1}{2}\cos^N(\tfrac{\pi}{N})$$