J01M.3

We have, in general,

$$\left(\frac{d\mathbf{L}}{dt}\right)_{space} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

where the *space* and *body* subscripts denotes the vector as seen in the space and body frames respectively, and $\boldsymbol{\omega}$ is the angular velocity of the rotating body frame. Here $\mathbf{L}$ can in general be any vector but if we take it to be angular momentum, then we have:

$$\mathbf{\Gamma}_{space} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} \tag{1}$$

where $\mathbf{\Gamma}$ is the torque exerted on the body as seen from the space frame.

If we choose the principal axes of the planet as our body frame, then $\mathbf{I}$ is diagonalized and we have:

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \begin{pmatrix} I_E & 0 & 0 \\ 0 & I_P & 0 \\ 0 & 0 & I_P \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Substituting into (1), we get:

$$\Gamma_x = I_E \dot{\omega}_1 + (I_P - I_P)\omega_2\omega_3 = I_E\dot{\omega}_1 \tag{2}$$

$$\Gamma_y = I_P\dot{\omega}_2 + (I_E - I_P)\omega_1\omega_3 = I_P\dot{\omega}_2 + (I_E - I_P)\omega_1\omega_3 \tag{3}$$

$$\Gamma_z = I_P\dot{\omega_3} + (I_P - I_E)\omega_1\omega_2 = I_P\dot{\omega_3} + (I_P - I_E)\omega_1\omega_2 \tag{4}$$

Therefore, for torque free precession where $\mathbf{\Gamma} = 0$, we have from (2), (3) and (4) respectively that:

$$\omega_1 = constant$$

$$I_P\dot{\omega_2} + (I_E - I_P)\omega_1\omega_3 = 0 \tag{5}$$

$$I_P\dot{\omega_3} + (I_P - I_E)\omega_1\omega_2 = 0 \tag{6}$$

If we define $p = \omega_2 + i\omega_3$, we can combine (5) and (6) into one equation

$$I_P\dot{p} + i\omega_1(I_P - I_E)p = 0 \tag{7}$$

$$\implies \int_{p_0}^{p} \frac{dp}{p} = -i\int_0^t \frac{\omega_1(I_P - I_E)}{I_P}\,dt$$

Performing the integral and re-arranging, we have:

$$p = p_0e^{-i\omega\epsilon t} \tag{8}$$

where we have made the substitution $\epsilon = \frac{I_P - I_E}{I_P}$.

Taking the real and imaginary parts of (8), we recover

$$\omega_2 = p_0\cos(\omega_1\epsilon t) \tag{9}$$

$$\omega_3 = -p_0\sin(\omega_1\epsilon t) \tag{10}$$

$\implies$ ω and hence the planet, is precessing at a frequency $\Omega = \omega_1\epsilon$.

| Assigned groups: ssgubser_classes_F2013

One thought on “J01M.3”

October 8, 2013 at 4:51 pm

Everything looks fine.