J01M.2

Solution to J01M.2 - Free Precession of a Planet 1

We first consider the forces on a small patch of the surface of the "spherical" object at the pole, letting this equal m_P:

$$\vec{F}_{pressure} - \frac{GMm_P}{r_P^2} = 0$$

where M is the mass of the planet as given in the problem and the pressure force acts on the patch with magnitude $F_{pressure} = \frac{P}{A_{patch}}$. For our purposes, we only need to know that this force will be the same at the pole and at the equator. This fact is assured by the spherical object being in hydrostatic equilibrium.

At the equator, we envision a similar patch m_E, which has a centripetal acceleration unlike our patch at the pole:

$$\vec{F}_{pressure} - \frac{GMm_E}{r_E^2} = m_E\omega^2 r_E$$

Substituting in $\vec{F}_{pressure}$ from the first equation, and dropping the masses of the patches (which we can take, for current purposes, to be equal), we get:

$$GM(\frac{1}{r_P^2} - \frac{1}{r_E^2}) = \omega^2 r_E$$
$$\Rightarrow \frac{GM}{r_P^2}(1 - \frac{r_P^2}{r_E^2}) = \omega^2 r_E$$

Now we use the condition $r_E = (1+\epsilon)r_P$ to express our equation in terms of ω, M, r_P:

$$GM(1-(1+\epsilon)^{-2}) = \omega^2 r_P^3(1+\epsilon)$$

$\Rightarrow GM(2\epsilon) = \omega^2 r_P^3(1+\epsilon)$ for small ϵ.

Then solving this equation:

$$\epsilon = \frac{\omega^2 r_P^3}{2GM - \omega^2 r_P^3} = \frac{1}{\frac{2GM}{\omega^2 r_P^3} - 1}$$

Alternate solution:

Instead of treating the forces on a patch of the surface, we consider the surface of the spinning planet to be an equipotential. Define the potential V(r, θ) as:

$$V(r,\theta) = -\frac{MG}{r} - \frac{1}{2}\omega^2 r^2 sin^2\theta$$

where M is the mass of the planet as given in the problem and the second term arises from considering the surface in the frame rotating with the planet. It is the centrifugal force that arises in such a frame. We have further ignored the other forces that arrive in the rotating frame by the assumption that $\dot{r} = 0$.

We then equate this potential at the pole (i.e. $\theta = 0$) to that at the equator ($\theta = \frac{\pi}{2}$):

$$\frac{MG}{r_P} = \frac{MG}{r_E} + \frac{1}{2}\omega^2 r_E^2$$

Using the condition that $r_E = (1+\epsilon)r_P$, we substitute this into our expression above and solve for ϵ:

$$\frac{MG}{r_P} = \frac{MG}{r_P(1+\epsilon)} + \frac{1}{2}\omega^2 (r_P(1+\epsilon))^2$$
$$\rightarrow MG(1+\epsilon) = MG + \frac{1}{2}\omega^2 r_P^3(1+\epsilon)^3$$

Now taking ϵ small, we use the binomial approximation to linear order for $(1+\epsilon)^3$ to find:

$$\epsilon = \frac{\omega^2 r_P^3}{2MG - 3\omega^2 r_P^3}$$

2 thoughts on “J01M.2”

The pressure vanishes at the surface of the planet for an obvious reason (otherwise it would push the outer infinitely small layer of the liquid outside the planet with an infinite acceleration). So it is not clear what you mean by $\vec{F}_{pressure}$.

I've updated the page with a solution from an in-class hydrodynamics review which avoids the problem of the mysterious force which would be counterbalancing gravitational attraction in the case of the "test patch" in the original solution.

I should clarify that originally $\vec{F}_{pressure}$ was meant to signify a purely radial force, which means the cross-sectional area it acts on does not go to 0 even for an infinitely thin boundary surface.
