J01M.1

We can parametrize the particle position using the azimuthal angle ϕ taken from the center of the torus, and the angle θ measured counterclockwise relative to the center of the small circle of radius b.

$z = b\sin(\theta)$,

$r=a-b \cos(\theta)$

which makes it clear that

$x = (a - b\cos(\theta))\cos(\phi)$and

$y = (a - b\cos(\theta))\sin(\phi)$ Using these coordinates, it is easy to write down our Lagrangian,

$$\mathcal{L} = \frac{m}{2} b^2 \dot{\theta}^2 + \frac{m}{2}(a - b\cos\theta)^2 \dot{\phi}^2 - mgb\sin\theta$$

This Lagrangian has no dependence on ϕ, so we have a conserved quantity (the angular momentum l_z). $l_z = m(a + b\cos(\phi))^2 \dot{\phi}$. Writing down the other Euler-Lagrange equation,

$$mb^2\ddot{\theta} - mb(a - b\cos\theta)\sin\theta\dot{\phi}^2 + mgb\cos\theta = 0.$$

Using l_z we can rewrite this as

$$mb^2 - \frac{b l_z^2 \sin\theta}{m(a - b\cos\theta)^3} + mgb\cos\theta = 0.$$

Now we wish to expand about the equilibrium angle ϕ_o where the particle would be in perfect circular motion. We write $\phi = \phi_o + \phi'$ where ϕ' is a small quantity. We also write $R_o = a - b\cos\theta_o$. After expanding we find

$$mb^2\ddot{\theta}' - \frac{bl_z^2}{m}\frac{(\sin\theta_o+\cos\theta_o\theta')}{a-b\cos\theta_o)^3}\left(1-\frac{3b\sin\theta_o\theta'}{a-b\cos\theta_o}\right)+mgb\cos\theta_o - mgb\sin\theta_o\theta' = 0 \ .$$

the equilibrium value θ_o is that which makes the constant term vanish. Then we are left with a harmonic oscillator equation $\ddot{\theta}' = -\omega^2\theta'$.

$$\ddot{\theta}' = -\frac{1}{mb^2}\left(-\frac{bl_z^2\cos\theta_O}{m(a-b\cos\theta_o)^3}+\frac{3b^2l_z^2\sin^2\theta_o}{m(a-b\cos\theta_o)^4}-mgb\sin\theta_o\right)\theta' + O(\theta'^2)$$

Hence $\omega^2 = \frac{1}{mb^2}\left(-\frac{bl_z^2\cos\theta_O}{m(a-b\cos\theta_o)^3}+\frac{3b^2l_z^2\sin^2\theta_o}{(a-b\cos\theta_o)^4}-mgb\sin\theta_o\right)$

Where $\frac{bl_z^2\sin\theta_o}{m(a-b\cos\theta_o)^3} = mgb\cos\theta_o$ determines θ_o.

Which yields $\cot\theta_o = \frac{\omega_o^2 R_o}{g}$ as $l_z = m\omega_o R_o^2$. Note that this is exactly the value that treating the case of circular motion using Newton's laws yields.

Substituting in this value we can simplify ω^2 to

$$\omega^2 = \frac{m\omega_o^2 R_o}{b}\cos\theta_o - 3m\omega_o^2\sin^2\theta_o + \frac{g}{b}\sin\theta_o$$

One thought on "J01M.1"

No, the term $\frac{1}{(a+b\cos\phi)^2}$ actually produces linear terms when expanded around ϕ_0. Why not?