We can parametrize the particle position using the azimuthal angle $\phi$ taken from the center of the torus, and the angle $\theta$ measured counterclockwise relative to the center of the small circle of radius $b$.

$$z = b \sin(\theta),$$

$$r = a - b \cos(\theta),$$

which makes it clear that

$$x = (a - b \cos(\theta)) \cos(\phi)$$

and

$$y = (a - b \cos(\theta)) \sin(\phi)$$

Using these coordinates, it is easy to write down our Lagrangian,

$$\mathcal{L} = \frac{m}{2} b^2 \dot{\theta}^2 + \frac{m}{2} (a - b \cos \theta)^2 \dot{\phi}^2 - mgb \sin \theta$$

This Lagrangian has no dependence on $\phi$, so we have a conserved quantity (the angular momentum $l_z$).

$$l_z = m(a + b \cos(\phi))^2 \dot{\phi}.$$ Writing down the other Euler-Lagrange equation,

$$mb^2 \ddot{\theta} - mb(a - b \cos \theta) \sin \theta \dot{\phi}^2 + mgb \cos \theta = 0.$$ 

Using $l_z$ we can rewrite this as

$$mb^2 \ddot{\theta} - \frac{bl_z^2 \sin \theta}{m(a - b \cos \theta)^3} + mgb \cos \theta = 0.$$ 

Now we wish to expand about the equilibrium angle $\phi_o$ where the particle would be in perfect circular motion. We write $\phi = \phi_o + \phi'$ where $\phi'$ is a small quantity. We also write $R_o = a - b \cos \theta_o$. After
expanding we find

\[ mb^2 \ddot{\theta}' = \frac{b l_z^2}{m} \left( \frac{\sin \theta_o + \cos \theta_o \theta'}{a - b \cos \theta_o} \right)^3 \left( 1 - \frac{3b \sin \theta_o \theta'}{a - b \cos \theta_o} \right) + m g b \cos \theta_o - m g b \sin \theta_o \theta' = 0. \]

the equilibrium value \( \theta_o \) is that which makes the constant term vanish. Then we are left with a harmonic oscillator equation \( \ddot{\theta}' = -\omega^2 \theta' \).

\[ \ddot{\theta}' = -\frac{1}{mb^2} \left( -\frac{b l_z^2 \cos \theta_o}{m(a - b \cos \theta_o)^3} + \frac{3b^2 l_z^2 \sin^2 \theta_o}{m(a - b \cos \theta_o)^4} - m g b \sin \theta_o \right) \theta' + O(\theta'^2) \]

Hence \( \omega^2 = \frac{1}{mb^2} \left( -\frac{b l_z^2 \cos \theta_o}{m(a - b \cos \theta_o)^3} + \frac{3b^2 l_z^2 \sin^2 \theta_o}{m(a - b \cos \theta_o)^4} - m g b \sin \theta_o \right) \)

Where \( \frac{b l_z^2 \sin \theta_o}{m(a - b \cos \theta_o)^3} = m g b \cos \theta_o \) determines \( \theta_o \).

Which yields \( \cot \theta_o = \frac{\omega_o^2 R_o}{g} \) as \( l_z = m \omega_o R_o^2 \). Note that this is exactly the value that treating the case of circular motion using Newton’s laws yields.

Substituting in this value we can simplify \( \omega^2 \) to

\[ \omega^2 = \frac{m \omega_o^2 R_o}{b} \cos \theta_o - 3m \omega_o^2 \sin^2 \theta_o + \frac{g}{b} \sin \theta_o \]
No, the term \( \frac{1}{(a+b \cos \phi)^2} \) actually produces linear terms when expanded around \( \phi_0 \). Why not?