Princeton University

J01M.1 - particle on a torus

Solution to J01M.1 — particle on a torus

Newtonian Solution

In uniform circular motion, we know there must be some angle α such that the net force on the particle is equal to $\frac{mv^2}{r}$ and pointing towards the axis of the torus, as shown in figure 1.

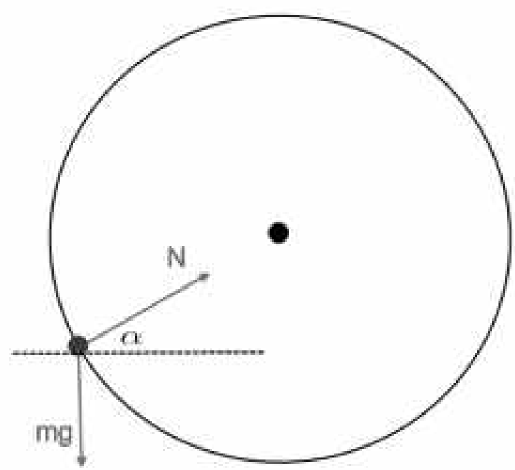

Figure 1: Particle (blue dot) at its angular equilibrium position for uniform circular motion about the vertical axis of the torus.

The two forces on the particle are gravity and the normal force. Newton's second law in the vertical and horizontal directions then give us:

$$N \sin \alpha = mg \quad \text{and} \quad N \cos \alpha = \frac{mv^2}{r} \tag{1}$$

Now suppose the particle is perturbed from its angular equilibrium position α by a small displacement $\Delta\alpha$. Then the magnitude of the normal force will be unchanged, but its direction changes, as shown in figure 2.

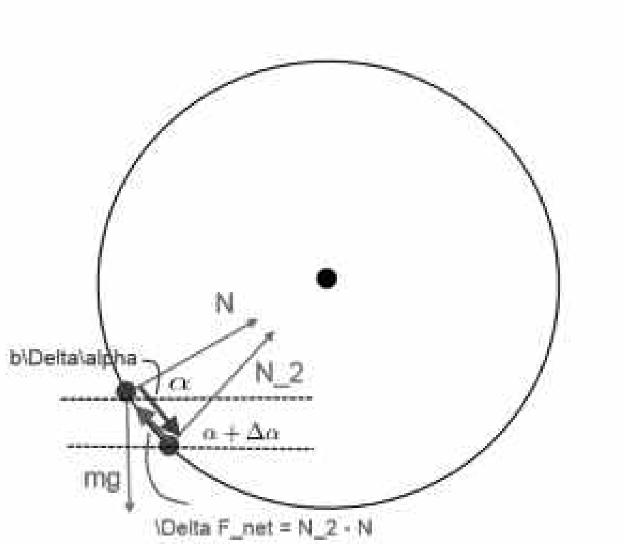

Figure 2: Forces before and after perturbing the particle by $\Delta\alpha$. There is a restoring force proportional to the small angular displacement.

The new normal force is given by

$$\vec{N}_2 = \begin{pmatrix} N \cos(\alpha + \Delta\alpha) \\ N \sin(\alpha + \Delta\alpha) \end{pmatrix} = N \begin{pmatrix} \cos\alpha \cos\Delta\alpha - \sin\alpha \sin\Delta\alpha \\ \cos\alpha \sin\Delta\alpha + \cos\Delta\alpha \sin\alpha \end{pmatrix} \tag{2}$$

$$\approx N \begin{pmatrix} \cos\alpha - \Delta\alpha \sin\alpha \\ \sin\alpha + \Delta\alpha \cos\alpha \end{pmatrix}$$

$$\vec{N}_2 = \vec{N} + \Delta\alpha \begin{pmatrix} mg \\ -\dfrac{mv^2}{r} \end{pmatrix} \tag{3}$$

Hence, $\vec{F}^{(\text{net})}_{\text{new}} = \vec{F}^{(\text{net})} + \Delta\vec{N}$ where $\Delta\vec{N}$ is the additional term in the equation above. Note that for small $\Delta\alpha$, $\Delta\vec{N}$ will point in the tangential direction and opposite the small angular displacement. This gives an equation of motion for $\Delta\alpha$ in addition to the uniform circular motion that existed before the perturbation:

$$mb\ddot{\Delta\alpha} = -\left|\Delta\vec{N}\right|\Delta\alpha \tag{4}$$

$$\ddot{\Delta\alpha} = -\frac{1}{mb}\sqrt{(mg)^2 + \left(\frac{mv^2}{r}\right)^2}\,\Delta\alpha \tag{5}$$

Which yields small oscillations with frequency given by

$$\omega^2 = \sqrt{\left(\frac{g}{b}\right)^2 + \left(\frac{v^2}{rb}\right)^2} \tag{6}$$

Moreover, dividing the two expressions in equation (1), we see that the equilibrium angle α is given by $\tan\alpha = \frac{rg}{v^2}$, so we can rewrite ω^2 as

$$\omega^2 = \frac{g}{b}\sqrt{1 + \left(\frac{v^2}{rg}\right)^2} = \frac{g}{b}\sqrt{1+\cot^2\alpha} = \frac{g}{b}\csc\alpha \tag{7}$$

Lagrangian Solution

Adopt the coordinates shown in figure 3.

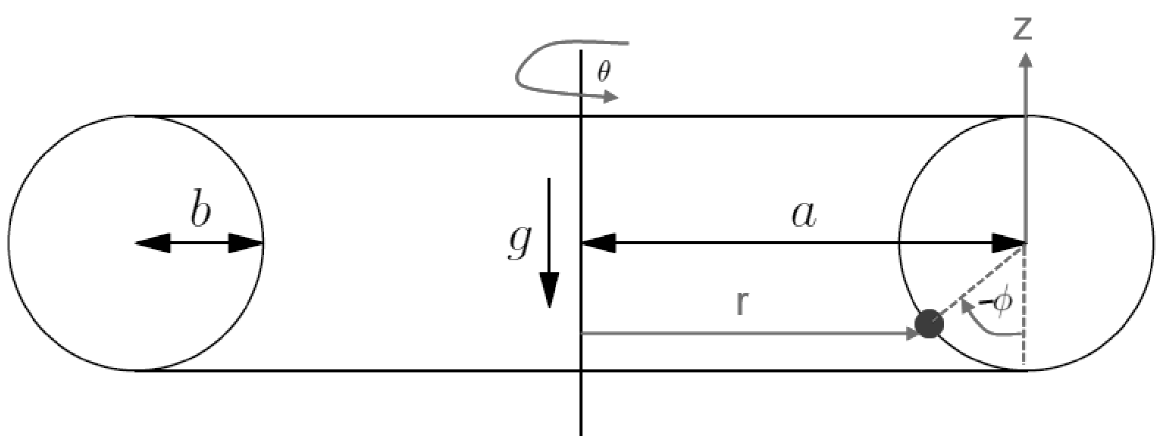

Figure 3: Coordinates used in this solution.

$$r = a + b\sin\phi \qquad z = -b\cos\phi \tag{8}$$

Then the Lagrangian is

$$\mathscr{L} = \frac{m}{2}\left(\dot{z}^2 + \dot{r}^2 + \left(r\dot{\theta}\right)^2 - 2gz\right) \tag{9}$$

$$\mathscr{L} = \frac{m}{2}\left(b^2\dot{\phi}^2 + r^2\dot{\theta}^2 + 2bg\cos\phi\right) \tag{10}$$

Lagrange's equations:

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{11}$$

$$b\ddot{\phi} - r\cos\phi\dot{\theta}^2 + g\sin\phi = 0 \tag{12}$$

For uniform circular motion (which we want to find small oscillations about), $\ddot{\phi} = 0$ and also $\ddot{\theta} = 0$. Hence in equilibrium, $\dot{\theta} = \Omega$. Reducing equation (12), we find the poloidal equilibrium angle ϕ_0 to be

$$0 = -r_0\cos\phi_0\Omega^2 + g\sin\phi_0 \quad \Rightarrow \quad \Omega^2 = \frac{g\tan\phi_0}{r_0} \tag{13}$$

Where r_0 is some equilibrium length $r_0 = a + b\sin\phi_0$. By equation (11), we know that $r^2\dot{\theta}$ is constant, so this constant must be $r_0^2\Omega$. Hence, we can write

$$\dot{\theta} = \left(\frac{r_0}{r}\right)^2\Omega \quad \Rightarrow \quad \dot{\theta}^2 = \left(\frac{r_0}{r}\right)^4\Omega^2 = \left(\frac{r_0}{r}\right)^3\frac{g\tan\phi_0}{r} \tag{14}$$

Substitute this expression for $\dot{\theta}^2$ into equation (12) to get

$$b\ddot{\phi} - g\tan\phi_0\left(\frac{r_0}{r}\right)^3\cos\phi + g\sin\phi = 0 \tag{15}$$

Now, suppose we are interested only in small oscillations of ϕ from its equilibrium position. Then we can write $\phi = \phi_0 + \Delta\phi$ and neglect $O(\Delta\phi^2)$ terms in our equation. F}irst, try to reduce $\frac{r_0}{r}$ since this should be close to 1.

$$\frac{r_0}{r} = \frac{r_0}{a + b\sin\phi} \approx \frac{r_0}{a + b\sin\phi_0 + b\Delta\phi\cos\phi_0} \tag{16}$$

$$= \frac{r_0}{r_0 + b\Delta\phi\cos\phi_0} = \frac{1}{1 + \frac{b\cos\phi_0}{a+b\sin\phi_0}\Delta\phi} \tag{17}$$

Note $\left|\frac{b\cos\phi_0}{a+b\sin\phi_0}\Delta\phi\right| < \frac{b}{a}|\Delta\phi| < \frac{1}{2}|\Delta\phi| \ll 1$ since the major radius must be more than twice the minor radius in order for the torus to have the topology shown in the diagram. Hence, we can approximate equation (17) with the leading term of a geometric series

$$\frac{r_0}{r} \approx 1 - \frac{b\cos\phi_0}{a+b\sin\phi_0}\Delta\phi \Rightarrow \left(\frac{r_0}{r}\right)^3 \approx 1 - 3\,\frac{b\cos\phi_0}{a+b\sin\phi_0}\Delta\phi \tag{18}$$

Noting that $\ddot{\phi} = \ddot{\Delta\phi}$ and using the above approximation, equation (12) is now

$$b\ddot{\Delta\phi} - g\tan\phi_0\left(1 - 3\,\frac{b\cos\phi_0}{a+b\sin\phi_0}\Delta\phi\right)\left(\cos\phi_0 - \Delta\phi\sin\phi_0\right) + g\left(\sin\phi_0 + \Delta\phi\cos\phi_0\right) = 0 \tag{19}$$

$$\ddot{\Delta\phi} + \frac{g}{b}\left(\frac{3b\cos\phi_0\sin\phi_0}{a+b\sin\phi_0} + \sin\phi_0\tan\phi_0 + \cos\phi_0\right)\Delta\phi = 0 \tag{20}$$

This is an equation of motion for small oscillations in $\Delta\phi$ with frequency

$$\omega^2 = \frac{g}{b}\sqrt{\frac{3b\cos\phi_0\sin\phi_0}{a+b\sin\phi_0} + \sin\phi_0\tan\phi_0 + \cos\phi_0} \tag{21}$$

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3 thoughts on “J01M.1 - particle on a torus”

Lagrangian solution looks fine, although I didn't check all of your computations.

In your Newtonian solution it is unclear why the magnitude of the normal force is unchanged.

Unfortunately, my Newtonian and Lagrangian solutions do not currently agree. They can be compared by noting that \alpha + \phi = \frac{\pi}{2}. Writing them in terms of the same angle shows that they differ only in the
\frac{3b\cos{\phi_0}\sin{\phi_0}}{a + b\sin{\phi_0}} term. This term arises from the small change in r due to the small oscillations in \Delta\phi. If instead \frac{r_0}{r} = 1, this term would vanish, as it does in the Newtonian solution. I think including this correction to r (and hence \omega^2) is more correct.