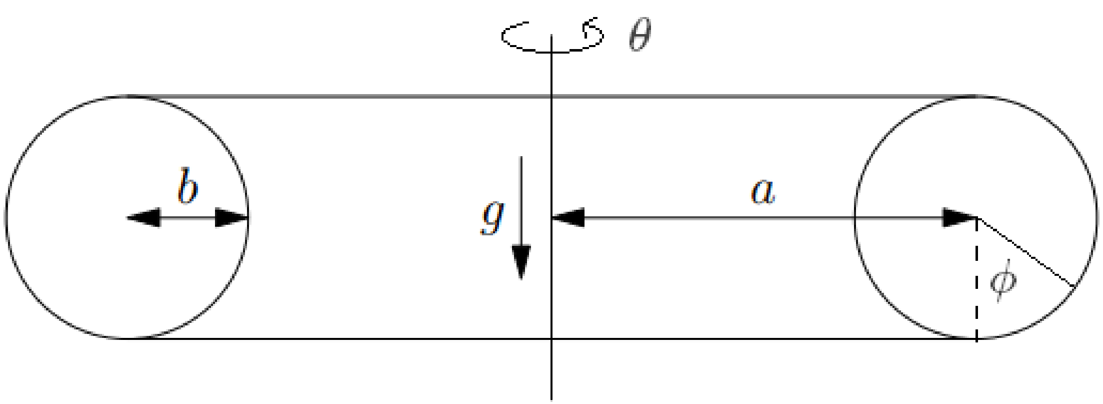
J01M.1

Problem

Find the frequency of small oscillations about uniform circular motion of a point mass that is constrained to move on the surface of a torus (donut) of major radius a and minor radius b whose axis is vertical.



Consider the circular cross section of the right side of the torus. Let ϕ be the angle the point mass makes with the vertical. Let r , z , and θ be the standard cylindrical coordinates of the point mass relative to the axis of the torus.

$$z = -b\cos\phi$$
$$\dot{z} = b\sin\phi\dot{\phi}$$
$$r = a + b\sin\phi$$
$$\dot{r} = b\cos\phi\dot{\phi}$$

Now we form the Lagrangian:

$$L = \tfrac{1}{2}m\dot{z}^2 + \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 - mgz$$

$$L = \frac{1}{2} b^2 \dot{\phi}^2 + \frac{1}{2} m(a + b\sin\phi)^2 \dot{\theta}^2 + gb\,\cos\phi$$

The Euler-Lagrange Equations for θ and ϕ.

(1) $m(a + b\sin\phi)\dot{\theta} = const = l$

(2) $b\ddot{\phi} = (a + b\sin\phi)\dot{\theta}^2 \cos\phi - g\sin\phi$

Equilibrium occurs when $\ddot{\phi} = 0$ and $\dot{\theta} = \Omega$. Then (2) gives the equilibrium angle ϕ_0 :

(3) $g\tan\phi_0 = (a + b\sin\phi_0)\Omega^2$

And (1) gives us a value for l

(4) $l = m(a + b\sin\phi_0)\Omega$

Eliminate $\dot{\theta}$ from (2) using (1)

$$b\ddot{\phi} = \frac{l^2}{m^2(a+b\sin\phi)} \cos\phi - g\sin\phi$$

Introduce a small displacement $\phi_0 \to \phi_0 + \epsilon$.

$$b\ddot{\epsilon} = \frac{l^2}{m^2\,(a+b\sin(\phi_0+\epsilon))} \cos(\phi_0 + \epsilon) - g\sin(\phi_0 + \epsilon)$$

Expanding the sines and cosines, then using the small angle approximation leads to:

$$b\ddot{\epsilon} = \frac{l^2}{m^2\,(a+b(\sin\phi_0+\epsilon\cos\phi_0))} (\cos\phi_0 - \epsilon\sin\phi_0) - g(\sin\phi_0 + \epsilon\cos\phi_0)$$

Taylor expanding the denominator

$$b\ddot{\epsilon} = \frac{l^2}{m^2\,(a+b\sin\phi_0)} (\cos\phi_0 - \epsilon\sin\phi_0 - \frac{b\epsilon\cos^2\phi_0}{a+b\sin\phi_0}) - g(\sin\phi_0 + \epsilon\cos\phi_0)$$

Observe that the terms independent of ϵ are the equilibrium condition and sum to 0 giving us:

$$b\ddot{\epsilon} = -(\frac{l^2}{m^2\,(a+b\sin\phi_0)} (\sin\phi_0 + \frac{b\epsilon\cos^2\phi_0}{a+b\sin\phi_0}) + g\cos\phi_0)\epsilon$$

Use (4) to eliminate l and then (3) to eliminate Ω

$$\ddot{\epsilon} = -\frac{g}{b} (\tan\phi_0(\sin\phi_0 + \frac{b\epsilon\cos^2\phi_0}{a+b\sin\phi_0}) + \cos\phi_0)\epsilon$$

Therefore the frequency of small oscillations is:

$$\omega^2 = \frac{g}{b} (\tan\phi_0(\sin\phi_0 + \frac{b\epsilon\cos^2\phi_0}{a+b\sin\phi_0}) + \cos\phi_0)$$

3 thoughts on "J01M.1"

OK.

Approach is correct. However something should be fixed.
Note that a particle on a torus has two degrees of freedom. But your Lagrangian has only one degree of freedom ϕ, because you substituted Ω for $\dot{\theta}$ (which is not legitimate). $\dot{\theta}$ is not a conserved quantity of the problem (it changes as ϕ changes). You indeed can eliminate $\dot{\theta}$, but not in such a way.

I have updated my solution to use angular momentum to eliminate $\dot{\theta}$
