J01E.3

Consider a single frequency ω in the three regions we have:

$$\begin{aligned} E_i = E_0 e^{i(wt-kz)} \ \ for\ z<0 \\ E_1 = E_1 e^{i(wt-k'z)} \ \ for\ 0<z<a \\ E_2 = E_2 e^{i(wt-kz)} \ \ for\ a<z \end{aligned} \tag{1}$$

where $k' = \frac{\omega n}{c} = \omega_0 + \frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0} (\omega - \omega_0)$.

From continuity of E parallel:

$$\begin{aligned} E_i(t,0) = E_1(t,0) \\ E_1 e^{-ik'a} = E_2 e^{-ika} \end{aligned} \tag{2}$$

Now we wish to work out the frequency profile of the wave, to do this we consider the wave at z=0

$$E(\omega) = \int \frac{dt}{\sqrt{2\pi}} E(t,0) e^{-i\omega t} \tag{3}$$

We can then match the different frequency components of the wave at the boundary, using the above boundary conditions, to deduce how the wave propagates in the medium (and by applying a factor of e^{-ikz} to calculate the waves spatial propagation).

Using this we can calculate the wave in the medium as follows:

$$E_1(t,z) = \int \frac{d\omega}{\sqrt{2\pi}} E_1(\omega) e^{i\omega t - ik'z} \tag{4}$$

$$E_1(t,z) = \int \frac{d\omega}{\sqrt{2\pi}} E_0(\omega) e^{i\omega t - ik'z}$$

$$E_1(t,z) = \int\int \frac{dt'}{\sqrt{2\pi}} \frac{d\omega}{\sqrt{2\pi}} E(t',0) e^{-i\omega t} e^{i\omega t - i\frac{(\omega_0 + \frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0}(\omega-\omega_0))}{c}z} \tag{5}$$

$$E_1(t,z) = \int dt' e^{-i\omega_0 (1-\frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0})z} E(t',0)\delta(t-t'-\frac{(\omega_0 + \frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0}(\omega-\omega_0))}{c}z) \tag{6}$$

\begin{equation}
E_1(t,z) = e^{-i\omega_0 (1-\frac{d\omega n (\omega)}{d\omega}\bigg|_{\omega_0})z} f(t-\frac{(\omega_0 + \frac{d\omega n (\omega)}{d\omega}\bigg|_{\omega_0} (\omega - \omega_0)) }{c}z)

\end{equation}

Thus the wave propagates with the same profile but now with a phase and a different velocity. An identical process can be performed for the wave after it has left the medium:

$$E_2(t,z) = \int \frac{d\omega}{\sqrt{2\pi}} E_2(\omega) e^{i\omega t - ikz} \tag{7}$$

$$E_2(t,z) = \int \frac{d\omega}{\sqrt{2\pi}} E_0(\omega) e^{i(k-k')a} e^{i\omega t - ikz}$$

$$E_2(t,z) = \int\int \frac{dt'}{\sqrt{2\pi}} \frac{d\omega}{\sqrt{2\pi}} E_0(t',0) e^{-i\omega t'} e^{i(k-k')a} e^{i\omega t - ikz}$$

$$E_2(t,z) = e^{-i\omega_0 (1-\frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0})a} f(t - \frac{\frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0}}{c} a - \frac{z}{c}) \tag{8}$$

When $\frac{d\omega n(\omega)}{d\omega}\Big|_{\omega_0}$ is less then zero (i.e. the group velocity is less than zero) the wave appears to jump in time!!