J01E.2

Solution to J01E.2 — Betatron

A big assumption we make here is to ignore the longitude acceleration of electron since in the limit case this acceleration ceases. The radiation therefore mainly comes from transversal acceleration. As suggested in the hint, this problem can be solved by writing out Newton's 2nd law. Thankfully the electron is constrain to move in a circle. In polar coordinate, the acceleration is simply

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} = (-r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta})\hat{\theta} \tag{1}$$

The $\hat{r}$ direction of force is supported by B field,

$$F_r = q(\mathbf{v} \times \mathbf{B})_r = qvB \tag{2}$$

Meanwhile the circular motion implies

$$m\gamma \frac{v^2}{R} = F_r = qvB \tag{3}$$

Using 3 to get rid of v, the r direction of force is

$$F_r = \frac{q^2 B^2 R}{m\gamma} \tag{4}$$

and the r direction of equation of motion is

$$\frac{q^2 B^2 R}{m^2\gamma^2} = R\dot{\theta}^2 \tag{5}$$

What about the $\hat{\theta}$ direction? The changing of B field induces a circular E field which accelerates electron in the same direction as its velocity. The induced E field can be found by using $EMF = -\frac{d\Phi}{dt}$

$$E_\theta = \frac{1}{2\pi R}\left[-\frac{d}{dt}\int_0^R dr(2\pi r)B(r,t)\right] = \frac{1}{2\pi R}\left[-\int_0^R dr(2\pi r)\frac{dB}{dt}\right] \tag{6}$$

$$= -\frac{R}{2}\frac{dB_{avg}}{dt} \tag{7}$$

Thus, the θ direction of motion is,

$$R\ddot{\theta} = \frac{qR}{2m\gamma}\frac{dB_{avg}}{dt} \tag{8}$$

By taking derivative of 5, we can obtain another expression of $\ddot{\theta}$ in terms of dB/dt.

$$\ddot{\theta} = \frac{2q^2 B}{m^2\gamma^2}\frac{dB}{dt}\frac{1}{2\dot{\theta}} = \frac{q}{m\gamma}\frac{dB}{dt} \tag{9}$$

Setting 8 and 9 equal,

$$2\frac{dB}{dt} = \frac{dB_{avg}}{dt} \tag{10}$$

Therefore we know $B = \frac{1}{2}B_{avg}$.

The max kinetic energy is achieved when the power of radiation equals to the power plumped in by E field. Larmor formula gives radiation in the rest frame of electron. Acceleration in right angle to velocity transforms according to $a' = a\gamma^2$. In lab frame,

$$P = \frac{\mu_0 a^2\gamma^4 q^2}{6\pi c} = \frac{\mu_0 q^4 B^2\gamma^2 c^2}{(6\pi c)m^2} \tag{11}$$

$$P = F_\theta c = q\frac{dB}{dt}Rc \tag{12}$$

Set them equal and solve for γ_{max} as a function of dB/dt, B, and R. Then plug in the energy equation.

$$E = mc^2\gamma_{max} = mc^2\left(\frac{6\pi R m^2}{\mu_0 q^3 B^2}\frac{dB}{dt}\right)^{1/2} \tag{13}$$

Plug in number we have $\gamma_{max} \approx 200$ and $E_{max} \approx 100$ MeV

One thought on "J01E.2"

December 13, 2013 at 12:31 pm

Okay, your answer $B = \frac{1}{2}B_{avg}$ is correct, but your derivation of it is slightly inaccurate. If you differentiate (5), you don't really get (9) because γ also depends on time (which you neglected). However, it didn't affect your result since the correct way to do it is to use $\frac{d(\gamma m R\dot{\theta})}{dt} \equiv \left(\frac{dp}{dt}\right)_{radial} = qE_\theta$, and so (5) actually implies $\frac{d}{dt}(\gamma m R\dot{\theta}) = qR\frac{dB}{dt}$, which then leads to the same result $B = \frac{1}{2}B_{avg}$.
Regarding your derivation of the maximal energy, it looks OK to me. However, one things needs more explanation in my opinion. You calculate the radiated power in the rest frame (which makes sense, as the formula $P \propto \ddot{d}^2$ with d the dipole moment works only for non-relativistic particles), but then use this result as if it is a radiated power in the lab frame. Why is it correct?
