1 January 2000, Thermodynamics, Problem 3

1.1 (a)

When $\rho < \rho_C$, the particles follow the usual Bose-Einstein distribution:

$$\bar{n}(k) = \frac{1}{e^{(E(k)-\mu)/k_BT} - 1} = \frac{1}{e^{(\hbar^2k^2/2m-\mu)/k_BT} - 1} \quad (1)$$

1.2 (b)

When $\rho > \rho_C$, things are not so simple any more. This is equivalent to $T < T_C$ when we hold N constant. Let's calculate it for that case; the only occupied states are the ground state and the first excited state. The chemical potential is greater than 0 but much smaller than 1, so that:

$$\bar{n}(k) = \frac{1}{e^{\hbar^2k^2/2mk_BT} - 1} \quad k \neq 0 \quad (2)$$

To find the occupancy of the ground state, find:

$$N_{excited} = \int \bar{n}(k)\, d^Dk$$

and then:

$$N_{ground} = N - N_{excited} = \rho V - N_{excited} \quad (3)$$

This happens because in the thermodynamic limit, $\mu \approx 0$ even <u>within</u> the condensate regime.

1.3 (c)

$$N_C = \int g(k) \frac{1}{e^{\hbar^2k^2/2mk_BT} - 1} d^Dk \approx \int g(k) \frac{1}{e^{\hbar^2k^2/2mk_BT} - 1} k^{D-1}\, dk$$

Changing variables to $x = \frac{\hbar^2k^2}{2mk_BT}$ we get a dimensionless integral times some constants times $T^{D/2}$. Thus:

$$\gamma = \frac{D}{2} \quad (4)$$

1.4 (d)

The release of particles causes a loss of temperature and hence a reduction in the critical density. Our goal is to find a condition such that the direct reduction of density is smaller than the reduction in the critical density (so that the system remains condensed):

$$\rho_C = bT^{3/2}$$

where b is a constant and we are assuming D=3.

$$\frac{d\rho_C}{\rho_C} = \frac{3}{2}\frac{dT}{T}$$

The internal energy of the system is:

$$U = \int g(E)\bar{n}(E)\,dE = \int \frac{Vm^{3/2}\sqrt{2E}}{2\pi^2}\frac{1}{e^{\hbar^2k^2/2mk_BT}-1}\,dE$$

A change of variables gives:

$$U = \frac{VT^{5/2}m^{3/2}}{2^{1/2}\pi^2}\int_0^\infty \frac{t^{3/2}}{e^t-1}\,dt$$

$$dU = \frac{5U}{2T}dT$$

This is the change in energy due to the temperature change. The change in energy due to the release of particles is:

$$dU = AedN = \frac{AU}{N}dN$$

assuming the energy remains roughly constant during the particle release (the release is small). Setting the two equal and multiplying the left hand side by V/V:

$$\frac{d\rho}{\rho} = \frac{5}{2A}\frac{dT}{T}$$

We want this to be less than the change of the critical density:

$$\frac{5}{2A} < \frac{3}{2}$$

$$A > \frac{5}{3} \tag{5}$$