January 2000 Preliminary Exam, Mechanics Problem 3

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Problem (Orbiting Mass on a String):

A mass m_1 slides without friction on a horizontal table. The mass is tied to a string with negligible mass that passes without friction through a small hole. A mass m_2 is tied to the other end of the string. The uniform gravitational acceleration g is normal to the table.

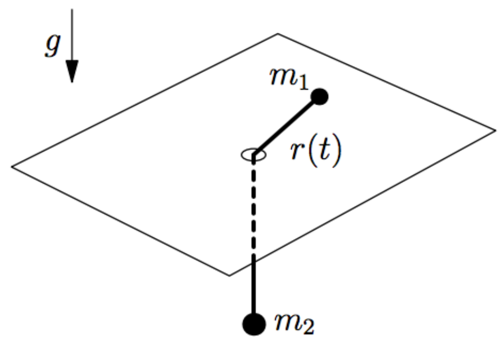

Figure 1: *Diagram from J00M.3*

The orbit of m_1 is only slightly perturbed from circular. The masses m_1 and m_2 are chosen so the orbit is closed, with one maximum and one minimum of the distance $r(t)$ of m_1 form the hole, when computed to first order in the departure from a circular orbit. Find m_2 in terms of the other parameters.

Reason for New Solution:

Alternate method used, with simpler notation.

Solution:

I will use coordinates r and ϕ, which define m_1's position on the table in polar coordinates. First, assemble the Lagrangian by analyzing each mass separately:

$$T_1 = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) \tag{1}$$

$$T_2 = \frac{1}{2}m_2\dot{r}^2 \tag{2}$$

$$U_1 = 0 \tag{3}$$

$$U_2 = m_2 g r \tag{4}$$

$$\implies L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m_2\dot{r}^2 - m_2 g r \tag{5}$$

ϕ is cyclic, so calculate the corresponding conserved quantity (angular momentum l):

$$\frac{\partial L}{\partial \dot{\phi}} = m_1 r^2 \dot{\phi} \equiv l \tag{6}$$

Now, use the Euler-Lagrange equation for r:

$$\frac{\partial L}{\partial r} = -m_2 g + m_1 r\dot{\phi}^2 = -m_2 g + \frac{l^2}{m_1 r^3} \tag{7}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) = m_1 \ddot{r} + m_2 \ddot{r} = (m_1 + m_2)\ddot{r} \tag{8}$$

$$\implies (m_1 + m_2)\ddot{r} = -m_2 g + \frac{l^2}{m_1 r^3} \tag{9}$$

Let r_0 be the equilibrium radius. At $r(t) = r_0$, $\ddot{r} = 0$, so

$$m_2 g = \frac{l^2}{m_1 r_0^3} \implies r_0 = (\frac{l^2}{m_1 m_2 g})^{1/3} \tag{10}$$

Perturb this radius such that $r(t) = r_0 + \epsilon(t) \implies \ddot{r} = \ddot{\epsilon}$. Then,

$$\begin{aligned} (m_1 + m_2)\ddot{\epsilon} &= \frac{l^2}{m_1 (r_0 + \epsilon)^3} - m_2 g \\ &\approx \frac{l^2}{m_1}(\frac{1}{r_0^3} - \frac{3}{r_0^4}\epsilon) - m_2 g \end{aligned} \tag{11}$$

$$\implies \ddot{\epsilon} = -\frac{3l^2}{m_1(m_1 + m_2)r_0^4}\epsilon \equiv -\omega_{osc}^2 \epsilon \tag{12}$$

This is a harmonic oscillator with frequency ω_{osc}, which must be the same frequency as the orbital frequency ω_{orb} of m_1 on the table top given that this orbital motion is closed with exactly one maximum and one minimum. Thus, ω_{orb} can be calculated using l as follows:

$$l = m_1 \omega_{orb} r_0^2 \implies \omega_{orb} = \frac{l}{m_1 r_0^2} \tag{13}$$

Therefore,

$$\omega_{orb}^2 = \omega_{osc}^2 \tag{14}$$

$$\implies \frac{l^2}{m_1^2 r_0^4} = \frac{3l^2}{m_1(m_1 + m_2)r_0^4} \tag{15}$$

$$\implies \frac{1}{m_1} = \frac{3}{m_1 + m_2} \tag{16}$$

$$\implies m_2 = 2m_1 \tag{17}$$