Seth Dorfman Prelims December 23, 2005

January 2000 CM

1) Consider a small section of arch of length Δx:

In the $\hat{x}$ direction

$$N(x)\cos(\theta(x)) - N(x+\Delta x)\cos(\theta(x+\Delta x)) = 0$$

$$\frac{N(x+\Delta x)\cos(\theta(x+\Delta x)) - N(x)\cos(\theta(x))}{\Delta x} = 0$$

$$\Rightarrow \frac{d}{dx}(N(x)\cos(\theta(x))) = 0$$

In the $\hat{y}$ direction:

$$N(x)\sin(\theta(x)) - N(x+\Delta x)\sin(\theta(x+\Delta x)) - m\Delta x g = 0$$

$$\Rightarrow \frac{d}{dx}(N(x)\sin(\theta(x))) = -mg$$

$$N(x)\sin(\theta(x)) = -mgx + C_1$$

At $x = \ell$ $N(x)\sin(\theta(x)) = 0$

$$\Rightarrow N(x)\sin(\theta(x)) = mg(\ell - x)$$

$$N(x)\cos(\theta(x)) = \cot(\theta(x))\, mg(\ell - x)$$

$$\frac{d}{dx}(N(x)\cos(\theta(x))) = \frac{d}{dx}(\cot(\theta(x))\, mg(\ell - x))$$

$$0 = \frac{d}{dx}(\cot(\theta(x))\, mg(\ell - x))$$

$$\cot(\theta(x)) = \frac{dx}{dy}$$

$$0 = \frac{d}{dx}\left(\frac{dx}{dy} mg(\ell - x)\right)$$

$$\frac{dx}{dy} mg(\ell - x) = C_2$$

$$\int mg(\ell - x)\,dx = \int C_2\,dy$$

$$mgx\left(\ell - \frac{x}{2}\right) = C_2 y + C_3$$

$x = 0, y = 0 \Rightarrow C_3 = 0$

$x = \ell, y = h \Rightarrow mg\,\ell\left(\ell - \frac{\ell}{2}\right) = C_2 h$

$$C_2 = \frac{mg}{h}\frac{\ell^2}{2}$$

$$\frac{mg}{h}\frac{\ell^2}{2} y = mgx\left(\ell - \frac{x}{2}\right)$$

$$y(x) = \frac{2h}{\ell^2} x\left(\ell - \frac{x}{2}\right)$$

$$y(x) = 2h\frac{x}{\ell}\left(1 - \frac{x}{2\ell}\right)$$